Primes in subsets and exponential sums
SSANT2021

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Prime Number Theorem (Hadamard, Vallée Poussin, 1896)

\[ \pi(x) := \sum_{p \leq x} 1 = \frac{x}{\log x} + O \left( \frac{x}{(\log x)^2} \right) \]

For primes in arithmetic progressions \( qr + a, \text{GCD}(a, q) = 1 \) the formula

\[ \pi(X; q, a) := \sum_{\substack{p \leq x \\text{ } \to p \equiv a \pmod{q}}} 1 = \frac{\pi(x)}{\varphi(q)} + o \left( \frac{\pi(x)}{\varphi(q)} \right) \]

is known:

- for all \( q \leq (\log X)^A \) as Siegel-Walfisz theorem (1936)
- for “almost all” \( q \leq X^{1/2-\varepsilon} \) as Bombieri-Vinogradov theorem (1965)
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Primes in subsets

Fractional part $\{x\} := x - \lfloor x \rfloor$

Define the set

$$\mathbb{E}(1/2) := \left\{ n \in \mathbb{N} : \{n^{1/2}\} < 0.5 \right\}$$

$n \in \mathbb{E}(1/2)$ means there is integer $k$ such that

$$k \leq n^{1/2} < (k + 0.5) \quad \text{or} \quad k^2 \leq n < (k + 0.5)^2$$

$$\sum_{\substack{p \leq x \\ p \in \mathbb{E}(1/2)}} 1 = \frac{1}{2} \pi(x) + O \left( x^{14/15} \right) \quad \text{Vinogradov (1940)}$$

Follows from the bound $\sum_{p \leq x} e(h\sqrt{p}) \ll X^{11/12}$
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Direction №1: reduce the domain for \( \{n^{1/2}\} \):

\[
A\left(\frac{1}{2}, c\right) := \left\{ n \in \mathbb{N} : \{n^{1/2}\} < n^{-c} \right\}
\]

Infinitely many primes in \( A(1/2, 1/2) \iff \) infinitely many primes of the form \( n^2 + 1 \). Currently the result is known for all \( c \leq 0.262 \) (Harman, Lewis 2001).

Direction №2: change the function \( \{n^{1/2}\} \rightarrow \{g(n)\} \). Consider \( g(n) = n^\alpha \):

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E(\alpha) := \left\{ n \in \mathbb{N} : \{n^\alpha\} < 0.5 \right\}
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If \( \alpha > 1, \alpha \notin \mathbb{N} \), the structure is more complicated: \( k^{1/\alpha} \leq n < (k + 0.5)^{1/\alpha} \)

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\sum_{\substack{p \leq x \ \text{p} \in E(\alpha)}} 1 = \frac{1}{2} \pi(x) + O\left(x^{1-\vartheta(\alpha)}\right) \quad \text{for all } \alpha > 0, \alpha \notin \mathbb{N} \quad \text{Vinogradov, Baker, Kolesnik, …}
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Primes from subsets in arithmetic progressions

One can establish the asymptotics for primes from $E(\alpha) \cap \{ qr + a \}$:

$$
\pi_{E(\alpha)}(x; q, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{q} \\ p \in E(\alpha)}} 1 = \frac{1}{2} \pi(x; q, a) + O \left( \frac{\pi(x; q, a)}{(\log x)^A} \right)
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As a corollary we obtain an analogue of Bombieri-Vinogradov theorem: for "almost all" $q \leq x^{\theta - \varepsilon}$:

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- $\alpha = 1/2$, $\theta = 1/4$, Tolev (1997)
- $1/2 \leq \alpha < 1$, $\theta = 1/3$, Gritsenko, Zinchenko (2013)
- $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\theta = 1/3$, S. (2019)
- $0 < \alpha < 1/9$, $\theta = 2/5 - 3\alpha/5$, S. (2020)
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Idea of the proof

The main goal

\[
\sum_{\begin{array}{c} p \leq x \\ p \equiv a \pmod{q} \end{array}} e(hp^\alpha) \ll \left( \frac{x}{q} \right)^{1-\delta}
\]

Tool №1: Van der Corput $k$-derivative test (due to Heath-Brown 2017):

\[
\sum_{n \sim y} e(g(n)) \ll y^{1+\varepsilon} \left( \left( g^{(k)}(y) \right)^{1/k(k-1)} + y^{-1/k(k-1)} + y^{-2/k(k-1)} g^{(k)}(y)^{-2/k^2(k-1)} \right)
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Tool №2: Vaughan & Heath-Brown identities to move from the sum over primes to the sum over all integers

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\sum_{p \leq x} e(hp^\alpha) \iff \sum_{n \leq y} e(Dn^\alpha)
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The larger $y$ is, the better
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Heath-Brown identity. Type I, II, III

Heath-Brown identity:

\[ \sum_{p \leq x, \ p \equiv a \ (\text{mod} \ q)} e(hp^\alpha) \rightarrow \sum \mu(d_{k+1}) \cdots \mu(d_{2k}) e\left( h(d_1 \ldots d_{2k})^\alpha \right) \]

The last is split into three types of sums. Fix the parameter \( \frac{1}{10} < \sigma < \frac{1}{6} \):

Type I situation: \( \exists \ d_i > x^{1/2+\sigma} \) (‘one long smooth variable’)

Type II situation: there is a partition \( \{d_1, \ldots, d_{2k}\} = S \cup T \):

\[ x^{1/2-\sigma} < \prod_{S} d_i \leq \prod_{T} d_i < x^{1/2+\sigma} \] (two long non-smooth variables)

Type III situation: \( \exists \ d_i \sim d_j \sim d_k \sim x^{1/3} \) (three smooth variables)

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Type I & II sums

Type I situation:

\[ \sum_{d_i \geq x^{1/2+\sigma}} e(Dd_i^\alpha) \quad \rightarrow \quad \text{Van der Corput k-test} \]

\[ d_i \equiv a \quad (\text{mod } q) \]

Type II situation:

\[ \sum_{x^{1/2-\sigma} < d_1 < x^{1/2+\sigma}} a(d_1) \quad \sum_{x^{1/2-\sigma} < d_2 < x^{1/2+\sigma}} b(d_2) e\left(D(d_1d_2)^\alpha\right) \quad \rightarrow \quad \text{Cauchy + Corput} \]

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\[d_1d_2 \equiv a \pmod{q} \]
Type III sum

Moving to characters

\[
\sum_{d_1, d_2, d_3 \sim x^{1/3}, \atop d_1 d_2 d_3 \equiv a \pmod{q}} e\left(D(d_1 d_2 d_3)^\alpha\right) \rightarrow
\]

\[
\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \sum_{d_1, d_2, d_3 \sim x^{1/3}} \chi(d_1) \chi(d_2) \chi(d_3) e\left(D(d_1 d_2 d_3)^\alpha\right)
\]

Apply Poisson summation + Stationary phase to (for example) \(d_2, d_3\):

\[
\left(\ldots\right) \sum_{d_1 \sim x^{1/3}} \sum_{s_2, s_3 \sim q/x^{1/3} - \alpha} |Kl_q(s_2, s_3)|
\]

Weil’s bound: \(|Kl_q(s_2, s_3)| \leq q^{1/2 + \varepsilon}(q, s_2, s_3)^{1/2} \).
Type III sum

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\[ \sum_{d_1, d_2, d_3 \sim x^{1/3}} e\left( D(d_1 d_2 d_3)^\alpha \right) \rightarrow \]

\[ \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \sum_{d_1, d_2, d_3 \sim x^{1/3}} \chi(d_1) \chi(d_2) \chi(d_3) e\left( D(d_1 d_2 d_3)^\alpha \right) \]

Apply Poisson summation + Stationary phase to (for example) \( d_2, d_3 \):

\[ (\ldots) \sum_{d_1 \sim x^{1/3}} \sum_{s_2, s_3 \sim q/x^{1/3} - \alpha} |Kl_q(s_2, s_3)| \]

Weil's bound: \( |Kl_q(s_2, s_3)| \leq q^{1/2+\epsilon} (q, s_2, s_3)^{1/2} \).
Type III sum

Moving to characters

\[
\sum_{d_1, d_2, d_3 \sim x^{1/3}} e \left( D(d_1 d_2 d_3)^\alpha \right) \rightarrow \\
\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \sum_{d_1, d_2, d_3 \sim x^{1/3}} \chi(d_1) \chi(d_2) \chi(d_3) e \left( D(d_1 d_2 d_3)^\alpha \right)
\]

Apply Poisson summation + Stationary phase to (for example) \(d_2, d_3\):

\[
\left( \ldots \right) \sum_{d_1 \sim x^{1/3}} \sum_{s_2, s_3 \sim q / x^{1/3 - \alpha}} |Kl_q(s_2, s_3)|
\]

Weil's bound: \(|Kl_q(s_2, s_3)| \leq q^{1/2 + \epsilon} (q, s_2, s_3)^{1/2} \).
Type III sum

Moving to characters

\[
\sum_{d_1, d_2, d_3 \sim x^{1/3}} \sum_{d_1 d_2 d_3 \equiv a \pmod{q}} e\left(D(d_1 d_2 d_3)^\alpha\right) \rightarrow
\]

\[
\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \sum_{d_1, d_2, d_3 \sim x^{1/3}} \chi(d_1) \chi(d_2) \chi(d_3) e\left(D(d_1 d_2 d_3)^\alpha\right)
\]

Apply Poisson summation + Stationary phase to (for example) \(d_2, d_3\):

\[
\left(\ldots\right) \sum_{d_1 \sim x^{1/3}} \sum_{s_2, s_3 \sim q/x^{1/3-\alpha}} |Kl_q(s_2, s_3)|
\]

Weil's bound: \(|Kl_q(s_2, s_3)| \leq q^{1/2+\varepsilon}(q, s_2, s_3)^{1/2} \).
Thank you for your attention!