Primes in subsets and exponential sums

SSANT2021

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Basic facts

Prime Number Theorem (Hadamard, Vallée Poussin, 1896)

$$\pi(x) := \sum_{p \leqslant x} 1 = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$$

For primes in arithmetic progressions $qr + a$, $\mathrm{GCD}(a, q) = 1$ the formula

$$\pi(X; q, a) := \sum_{\substack{p \leqslant x \\ p \equiv a \pmod q}} 1 = \frac{\pi(x)}{\varphi(q)} + o\left(\frac{\pi(x)}{\varphi(q)}\right)$$

is known:
-for all $q \leqslant (\log X)^A$ as Siegel-Walfisz theorem (1936)
-for "almost all" $q \leqslant X^{1/2-\varepsilon}$ as Bombieri-Vinogradov theorem (1965)

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Primes in subsets

Fractional part $\{x\} := x - \lfloor x \rfloor$

Define the set

$$\mathbb{E}(1/2) := \left\{ n \in \mathbb{N} : \{n^{1/2}\} < 0.5 \right\}$$

$n \in \mathbb{E}(1/2)$ means there is integer k such that

$$k \leqslant n^{1/2} < (k + 0.5) \qquad \text{or} \qquad k^2 \leqslant n < (k + 0.5)^2$$

$$\sum_{\substack{p \leqslant x \\ p \in \mathbb{E}(1/2)}} 1 = \frac{1}{2}\pi(x) + O\left(x^{14/15}\right) \qquad \textit{Vinogradov } (1940)$$

Follows from the bound $\sum_{p \leqslant x} e(h\sqrt{p}) \ll X^{11/12}$

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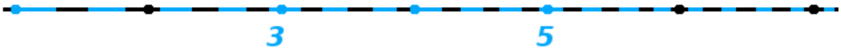
$$\mathbb{A}(1/2, c) := \left\{ n \in \mathbb{N} : \{n^{1/2}\} < n^{-c} \right\}$$

Infinitely many primes in $\mathbb{A}(1/2, 1/2) \iff$ infinitely many primes of the form $n^2 + 1$.
Currently the result is known for all $c \leqslant 0.262$ (Harman, Lewis 2001).

Direction №2: change the function $\{n^{1/2}\} \longrightarrow \{g(n)\}$. Consider $g(n) = n^{\alpha}$:

$$\mathbb{E}(\alpha) := \left\{ n \in \mathbb{N} : \{n^{\alpha}\} < 0.5 \right\}$$

If $\alpha > 1, \alpha \notin \mathbb{N}$, the structure is more complicated: $k^{1/\alpha} \leqslant n < (k + 0.5)^{1/\alpha}$



$$\sum_{\substack{p \leqslant x \\ p \in \mathbb{E}(\alpha)}} 1 = \frac{1}{2}\pi(x) + O\left(x^{1-\vartheta(\alpha)}\right)$$ for all $\alpha > 0, \alpha \notin \mathbb{N}$ *Vinogradov, Baker, Kolesnik, ...*

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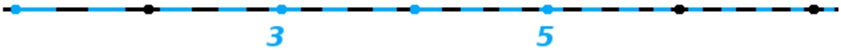

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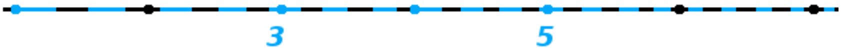

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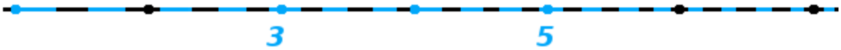

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Primes from subsets in arithmetic progressions

One can establish the asymptotics for primes from $\mathbb{E}(\alpha) \cap \{qr + a\}$:

$$\pi_{\mathbb{E}(\alpha)}(x; q, a) := \sum_{\substack{p \leqslant x \\ p \in \mathbb{E}(\alpha) \\ p \equiv a \pmod q}} 1 = \frac{1}{2}\pi(x; q, a) + O\left(\frac{\pi(x; q, a)}{(\log x)^A}\right)$$

As a corollary we obtain an analogue of Bombieri-Vinogradov theorem: for "almost all" $q \leqslant x^{\theta - \varepsilon}$:

$$\pi_{\mathbb{E}(\alpha)}(x; q, a) = \frac{\pi_{\mathbb{E}(\alpha)}(x)}{\varphi(q)} + O\left(\frac{\pi_{\mathbb{E}(\alpha)}(x)}{(\log x)^A}\right)$$

$\alpha = 1/2,$	$\theta = 1/4,$	Tolev (1997)
$1/2 \leqslant \alpha < 1,$	$\theta = 1/3,$	Gritsenko, Zinchenko (2013)
$\alpha > 0, \alpha \notin \mathbb{N}$	$\theta = 1/3,$	S. (2019)
$0 < \alpha < 1/9$	$\theta = 2/5 - 3\alpha/5$	S. (2020)

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Idea of the proof

The main goal

$$\sum_{\substack{p\leqslant x \\ p\equiv a \pmod q}} e(hp^{\alpha}) \ll \left(\frac{x}{q}\right)^{1-\delta}$$

Tool №1: Van der Corput k-derivative test (due to Heath-Brown 2017):

$$\sum_{n\sim y} e(g(n)) \ll y^{1+\varepsilon}\left(\left(g^{(k)}(y)\right)^{1/k(k-1)} + y^{-1/k(k-1)} + y^{-2/k(k-1)}\left(g^{(k)}(y)\right)^{-2/k^2(k-1)}\right)$$

Tool №2: Vaughan & Heath-Brown identities to move from the sum over primes to the sum over all integers

$$\sum_{p\leqslant x} e(hp^{\alpha}) \iff \sum_{n\leqslant y} e(Dn^{\alpha})$$

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Heath-Brown identity. Type I, II, III

Heath-Brown identity:

$$\sum_{\substack{p \leqslant x \\ p \equiv a \pmod q}} e(hp^\alpha) \longrightarrow \sum_{\substack{d_1 \ldots d_{2k} \sim x \\ d_1 \ldots d_{2k} \equiv a \pmod q \\ d_{k+1}, \ldots, d_{2k} < x^{1/k}}} \mu(d_{k+1}) \ldots \mu(d_{2k}) e\Big(h(d_1 \ldots d_{2k})^\alpha\Big)$$

The last is split into three types of sums. Fix the parameter $1/10 < \sigma < 1/6$:

Type I situation: $\exists\, d_i > x^{1/2+\sigma}$ ('one long smooth variable')

Type II situation: there is a partition $\{d_1, \ldots, d_{2k}\} = \mathbb{S} \cup \mathbb{T}$:

$$x^{1/2-\sigma} < \prod_{\mathbb{S}} d_i \leqslant \prod_{\mathbb{T}} d_i < x^{1/2+\sigma} \quad \text{(two long non-smooth variables)}$$

Type III situation: $\exists\, d_i \sim d_j \sim d_k \sim x^{1/3}$ (three smooth variables)

If $\sigma = 1/6$ Type III does not occur

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Type I & II sums

Type I situation:

$$\sum_{\substack{d_i \geqslant x^{1/2+\sigma} \\ d_i \equiv a \pmod q}} e\left(D d_i^{\alpha}\right) \longrightarrow \text{Van der Corput k-test}$$

Type II situation:

$$\sum_{x^{1/2-\sigma} < d_1 < x^{1/2+\sigma}} a(d_1) \sum_{\substack{x^{1/2-\sigma} < d_2 < x^{1/2+\sigma} \\ d_1 d_2 \approx x \\ d_1 d_2 \equiv a \pmod q}} b(d_2) e\Big(D (d_1 d_2)^{\alpha}\Big) \longrightarrow \text{Cauchy} + \text{Corput}$$

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Type III sum

Moving to characters

$$\sum_{\substack{d_1,d_2,d_3\sim x^{1/3}\\ d_1d_2d_3\equiv a \pmod q}} e\Big(D(d_1d_2d_3)^\alpha\Big) \longrightarrow$$

$$\frac{1}{\varphi(q)}\sum_{\chi \pmod q}\sum_{d_1,d_2,d_3\sim x^{1/3}} \chi(d_1)\chi(d_2)\chi(d_3)e\Big(D(d_1d_2d_3)^\alpha\Big)$$

Apply Poisson summation + Stationery phase to (for example) d_2, d_3:

$$\Big(\dots\Big)\sum_{d_1\sim x^{1/3}}\sum_{s_2,s_3\sim q/x^{1/3-\alpha}}\Big|\mathrm{Kl}_q(s_2,s_3)\Big|$$

Weil's bound: $|\mathrm{Kl}_q(s_2,s_3)| \leqslant q^{1/2+\varepsilon}(q,s_2,s_3)^{1/2}$.

Type III sum

Moving to characters

$$\sum_{\substack{d_1,d_2,d_3\sim x^{1/3}\\ d_1d_2d_3\equiv a \pmod q}} e\Big(D(d_1d_2d_3)^\alpha\Big) \longrightarrow$$

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Thanks

Thank you for your attention!