Correcting the subconvexity bound

- We want to obtain the bound
  \[ \sum_{n \sim \sqrt{T}} n^{it} \ll T^{1/2-\eta} \]
  with \( t \sim T \).

- We split into short intervals
  \[ \sum_{n \in [N_k, N_k + H]} n^{it} = N^{it} \sum_{h \in [0, H]} e \left( \frac{ht}{N} - h^2 \cdot \frac{t}{2N^2} + \ldots \right) \]
  with \( N_k = kH \sim \sqrt{T} \).
Overall strategy

Whenever

\[ \frac{t}{2N^2} \approx \frac{a}{q} + O\left(\frac{1}{H^2}\right) \]

with \( H^\delta < q < H^{2-\delta} \) we will exhibit cancellations in the short sum.

Whenever

\[ \frac{t}{2N^2} \approx \frac{a}{q} + O\left(\frac{1}{H^{2-\delta}}\right) \]

with \( q \leq H^\delta \) we will bound the sum trivially, but we will show there are few such intervals \([N, N + H]\).
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- Usually
  \[
  \sum_{h \in [0,H]} e(\varphi(h) + \theta(h)) \approx \sum_{h \in [0,H]} e(\varphi(h))
  \]
  for any \( \theta(h) \) with \( \theta'(h) \ll 1/H \) for \( h \in [0,H] \).

- We approximate
  \[
  \frac{t}{2N^2} = \frac{a}{q} + \theta_N
  \]
  with \( (a, q) = 1 \) and \( q \leq Q := H^{2-\delta} \) and \( |\theta_N| \leq 1/(qQ) \).

- As long as \( q > H^{\delta} \) we have \( |\theta_N| \leq 1/H^2 \) and this means that \( e(h^2\theta) \) can be ignored for \( q > H^{\delta} \).
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▶ Therefore for \( q > H^\delta \) we have

\[
\sum_{h \in [0,H]} e\left(\frac{ht}{N} - h^2 \left(\frac{t}{2N^2}\right)\right) \approx \sum_{h \in [0,H]} e\left(\frac{ht}{N} - \frac{h^2 a}{q}\right)
\]

▶ Furthermore given \( q \) we can find a \( b \) such that,

\[
\frac{t}{N} = \frac{b}{q} + \theta \pmod{1}
\]

with \(|\theta| \leq 1/q\).

▶ So we get that the short sum is

\[
\sum_{h \in [0,H]} e\left(\frac{h b}{q} - \frac{h^2 a}{q} + h\theta\right)
\]

with \(|\theta| \leq 1/q\).
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We now apply Poisson summation:

$$\sum_{h \in [0, H]} e\left(\frac{hb}{q} - \frac{h^2 a}{q} + h\theta\right) \approx \frac{1}{q} \sum_{|\ell| \leq q/H} S(b - \ell, a) 1\left(H\left(\theta - \frac{\ell}{q}\right)\right)$$

where

$$S(a - \ell, b) = \sum_{x \pmod{q}} e\left(\frac{x(a - \ell)}{q} + \frac{x^2 b}{q}\right) \ll \sqrt{q}$$

In particular bounding the right hand side trivially we get

$$\sum_{h \in [0, H]} e\left(\frac{hb}{q} - \frac{h^2 a}{q} + h\theta\right) \ll \frac{\sqrt{q}}{H} \leq H^{1-\delta/2}$$

since \(q \leq H^{2-\delta}\).
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This gives the first claim: that if

\[ \frac{t}{2N^2} = \frac{a}{q} + O\left(\frac{1}{H^2}\right) \]

with \( H^\delta \leq q \leq H^{2-\delta} \) then the short sum over \([N, N + H]\) is bounded by \( H^{1-\delta/2} \).

It remains to show that the number of intervals \([N, N + H]\) with

\[ \frac{t}{2N^2} = \frac{a}{q} + O\left(\frac{1}{H^{2-\delta}}\right) \]

and \( q \leq H^\delta \) is small.
Correcting the subconvexity bound

- We wish to show,

\[ \sum_{kH \sim \sqrt{T}} 1(\exists q \leq H^{\delta} : \left\| \frac{t}{2(kH)^2} - \frac{a}{q} \right\| \leq \frac{1}{H^{2-\delta}}) \ll \frac{\sqrt{T}}{H} \cdot H^{-\eta} \]

for some \( \eta > 0 \).

- We can drop the \( \exists \) by using the union bound. Bounding the above by

\[ \sum_{q \leq H^{\delta}} \sum_{\substack{kH \sim \sqrt{T} \\ (a,q)=1}} 1(\left\| \frac{t}{2(kH)^2} - \frac{a}{q} \right\| \leq \frac{1}{H^{2-\delta}}) \]

- In particular it’s enough to show

\[ \sum_{kH \sim \sqrt{T}} 1(\left\| \frac{t}{2(kH)^2} - \frac{a}{q} \right\| \leq \frac{1}{H^{2-\delta}}) \ll \frac{\sqrt{T}}{H} H^{-2\delta-\eta} \]

for some \( \eta > 0 \).
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- As usual we expand into a trigonometric series

\[ 1 \left( \left\| \frac{t}{2(kH)^2} - \frac{a}{q} \right\| \leq \frac{1}{H^{2-\delta}} \right) \approx \frac{1}{H^{2-\delta}} + \sum_{0 < |\ell| \leq H^{2-\delta}} e\left( \frac{\ell t}{2(kH)^2} \right) \]

- The main term is

\[ \frac{\sqrt{T}}{H^{3-\delta}} \ll \frac{\sqrt{T}}{H} H^{-2\delta-\eta} \]

for some \( \eta > 0 \), provided that \( \delta \) is sufficiently small.

- The error term is

\[ \frac{1}{H^{2-\delta}} \sum_{|\ell| \leq H^{2-\delta}} \sum_{k \sim \sqrt{T}/H} e\left( \frac{\ell t}{2k^2 H^2} \right) \]
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We apply Poisson summation in $k$. The new length is

$$
\varphi'(\frac{\sqrt{T}}{H}) \approx \frac{H^{2-\delta}T}{(\sqrt{T}/H)^3H^2} \approx \frac{H^{3-\delta}}{\sqrt{T}}
$$

If $H$ is sufficiently small power of $T$ then this is $< 1$. This means that only the central term survives and therefore the behavior of the sum is exactly the sum as the integral

$$
\int_{x \sim \sqrt{T}/H} e\left(\frac{\ell t}{2x^2H^2}\right)dx \ll \frac{\sqrt{T}}{H^{3-\delta}}
$$

by the first derivative test.

This is exactly the same bound as we obtained from the main term.
Correcting the subconvexity bound

- To summarize: we split the sum
  \[ \sum_{n \sim \sqrt{T}} n^t \]
  into \( \sqrt{T}/H \) intervals of length \( H \).

- If on the interval \([N, N + H]\) we have,
  \[ \frac{t}{2N^2} = \frac{a}{q} + O\left(\frac{1}{H^2}\right) \]
  for some \( H^\delta \leq H^{2-\delta} \), then we can bound the contribution of this interval by \( H^{1-\delta/2} \).

- The number of remaining intervals is (provided that \( H \) is choosen a small power),
  \[ \ll \frac{\sqrt{T}}{H^{3-\delta}} H^{2\delta} \]
  and this is less than \( \sqrt{T} H^{-1-\eta} \) for some \( \eta > 0 \) provided that \( \delta \) is sufficiently small.
Correcting the subconvexity

- These two together give us a subconvex bound for the Riemann zeta-function.

- If you go through the proof carefully you see that we also get an algorithm for computing the Riemann zeta function in time $O(T^{1/2-\delta})$ for some $\delta > 0$. 
Consequences of bounds for $\zeta(s)$

- We established a subconvex bound

\[ |\zeta(\frac{1}{2} + it)| \ll (1 + |t|)^{1/4 - \delta} \]

for some $\delta > 0$.

- In reality we expect

\[ |\zeta(\frac{1}{2} + it)| \ll_\varepsilon (1 + |t|)^\varepsilon \]

for any given $\varepsilon > 0$.

- This is called the Lindelof Hypothesis and is the strongest bound that one can hope for (at the scale of power-savings).
Lindelof Hypothesis

- It is easy to motivate the Lindelof Hypothesis: We can write

\[ \zeta(\frac{1}{2} + it) \approx \sum_{n \leq T} \frac{1}{n^{1/2+it}} \]

- Since the frequency \( p^it \) are uncorrelated it is reasonable to think of \( n^it \) as a random multiplicative function \( X_n \) with \( X_p \) uniformly distributed on the unit circle.

- We have,

\[ \sum_{n \leq T} \frac{X_n}{\sqrt{n}} \ll T^\varepsilon \]

with very high probability
Lindelof Hypothesis

- Alternatively we can compute the so-called second moment,

\[ \int_{|t| \leq T} |\zeta(\frac{1}{2} + it)|^2 \approx \int_{|t| \leq T} \left| \sum_{n \leq T} \frac{1}{n^{1/2 + it}} \right|^2 dt \sim T \sum_{n \leq T} \frac{1}{n} \]

- Which shows that for typical \( t \)

\[ |\zeta(\frac{1}{2} + it)| \ll \log T \]

- This is somewhat misleading though, because this bound is not true for all \( t \in [T, 2T] \). It is too optimistic.
Consequences of the Lindelof Hypothesis

- If the Lindelof Hypothesis is true then we can approximate the ζ function by short Dirichlet polynomials.

- We have, for σ > \( \frac{1}{2} \) and \( \mathcal{W} \) a smooth compactly supported function with \( \mathcal{W}(0) = 1 \),

\[
\sum_{n \leq X} \frac{1}{n^{\sigma + it}} \mathcal{W} \left( \frac{n}{X} \right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(\sigma + it + w) \tilde{\mathcal{W}}(w) X^w \, dw
\]

- Shifting contours to \( \Re s = \frac{1}{2} - \sigma \) we get

\[
\zeta(s) + \frac{1}{2\pi i} \int_{\frac{1}{2} - \sigma - i\infty}^{\frac{1}{2} - \sigma + i\infty} \zeta(\sigma + it + w) \tilde{\mathcal{W}}(w) X^w \, dw
\]

- On the Lindelof Hypothesis we can bound the integral by \( \ll_{\varepsilon} T^\varepsilon X^{1/2-\sigma} \) which is negligible as soon as \( X > T^{10\varepsilon} \), say.
Consequence of the Lindelof Hypothesis

Theorem

Assume the Lindelof Hypothesis. Let $\varepsilon > 0$ be given. Then, for $\sigma > \frac{1}{2}$ and $t \in [T, 2T],$

$$\zeta(\sigma + it) = \sum_{n \leq T^\varepsilon} \frac{1}{n^{1/2+it}} + O(T^{-(\sigma-1/2)\varepsilon}).$$

In other words on the Lindelof Hypothesis we can compute the Riemann zeta-function off the critical line in time $O(T^\varepsilon).$
Consequences of the Lindelof Hypothesis

- Another essentially similar consequence of the Lindelof Hypothesis is the bound
  \[
  \sum_{n \leq N} n^{it} \ll \varepsilon |t|^\varepsilon \sqrt{N}
  \]

- We will later see what these bounds have to say about the zeros of the Riemann \( \zeta \)-function.

- In particular we will discuss so called zero-density theorems: theorems that establish bounds for the number of points \( \beta + i\gamma \) with
  \[
  \zeta(\beta + i\gamma) = 0, \quad \beta > \sigma, \quad |\gamma| \leq T.
  \]

- The trivial bound is \( \ll T \log T \) since this is the total number of zeros with \( |\gamma| \leq T \).
Mollifiers

► A mollifier is a finite Dirichlet polynomial with the property that “most of the time”

\[ \zeta(s)M(s) \approx 1 \]

► Since

\[ \frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s} \]

where \( \mu \) is the Mobius function, it is natural to expect that

\[ \sum_{n \leq N} \frac{\mu(n)}{n^s} \]

should be a mollifier.

► By Mobius inversion

\[ \zeta(s)M(s) = 1 + \sum_{n > N} \frac{a(n)}{n^s} \]

where \( |a(n)| \leq d(n) \ll \varepsilon n^\varepsilon \).
Mollifiers

- We are interested in bounding

\[ N(\sigma; T) := \{ \beta + i\gamma : \zeta (\beta + i\gamma) = 0, \beta > \sigma, |\gamma| \leq T \} . \]

- Trivially

\[
N(\sigma; T) \leq \frac{1}{\varepsilon} \sum_{\beta > \sigma - \varepsilon} (\beta - \frac{1}{2}) \\
\sum_{|\gamma| \leq T} (M\zeta) (\beta + i\gamma) = 0
\]

- By Littewood’s formula (an analogue of Jensen formula for rectangles) the above is

\[
\leq \frac{1}{\varepsilon} \int_{|t| \leq T} \log |(\zeta M)(\sigma - \varepsilon + it)| dt
\]
Mollifiers

- By Jensen’s inequality

\[ \int_{|t| \leq T} \log |(M \zeta)(\sigma + it)| \, dt \leq T \log \left( \frac{1}{2T} \int_{|t| \leq T} |(\zeta M)(\sigma + it)|^2 \, dt \right) \]

- One typically simply computes the above second moment, and this is the “classical” way of obtaining a zero-density estimate.

- For instance if we \( M \) is a mollifier of length \( X \), then we expect

\[ \frac{1}{2T} \int_{|t| \leq T} |(\zeta M)(\sigma + it)|^2 \, dt = 1 + O \left( \sum_{n > X} \frac{1}{n^{2\sigma}} \right) = 1 + O(X^{-(2\sigma-1)}) \]
Mollifiers

And this would lead to a zero density theorem of the form,

\[ N(\sigma, T) \ll TX^{-(2\sigma-1)} \]

In practice it is fairly easy to compute such expressions with \( X = T^{1/2} \) and this leads to a standard zero-density bound of the form

\[ N(\sigma, T) \ll \frac{1}{\varepsilon} \cdot TX^{-(2\sigma-1+2\varepsilon)} \]

A reasonable choice of \( \varepsilon \) is \( 1/\log T \) and this would then give

\[ \ll T^{1-(\sigma-\frac{1}{2})\log T} \]