Correcting the subconvexity bound

- We want to obtain the bound

$$\sum_{n\sim\sqrt{T}} n^{it} \ll T^{1/2-\eta}$$

with $t \sim T$.

- We split into short intervals

$$\sum_{n\in[N_k,N_k+H]} n^{it} = N^{it} \sum_{h\in[0,H]} e\Big(\frac{ht}{N} - h^2 \cdot \frac{t}{2N^2} + \dots\Big)$$

with $N_k = kH \sim \sqrt{T}$.

Overall strategy

- Whenever
$$\frac{t}{2N^2} \approx \frac{a}{q} + O\Big(\frac{1}{H^2}\Big)$$
with $H^\delta < q < H^{2-\delta}$ we will exhibit cancellations in the short sum
- Whenever
$$\frac{t}{2N^2} \approx \frac{a}{q} + O\Big(\frac{1}{H^{2-\delta}}\Big)$$
with $q \leq H^\delta$ we will bound the sum trivially, but we will show there are few such intervals $[N, N+H]$.

Correcting the subconvexity bound

- Usually

$$\sum_{h\in[0,H]} e(\varphi(h)+\theta(h)) \approx \sum_{h\in[0,H]} e(\varphi(h))$$

for any $\theta(h)$ with $\theta'(h) \ll 1/H$ for $h \in [0,H]$.

- We approximate

$$\frac{t}{2N^2} = \frac{a}{q} + \theta_N$$

with $(a,q)=1$ and $q \leq Q := H^{2-\delta}$ and $|\theta_N| \leq 1/(qQ)$.

- As long as $q > H^\delta$ we have $|\theta_N| \leq 1/H^2$ and this means that $e(h^2\theta)$ can be ignored for $q > H^\delta$.

Correcting the subconvexity bound

- Therefore for $q > H^\delta$ we have

$$\sum_{h\in[0,H]} e\Big(\frac{ht}{N} - h^2\Big(\frac{t}{2N^2}\Big)\Big) \approx \sum_{h\in[0,H]} e\Big(\frac{ht}{N} - \frac{h^2 a}{q}\Big)$$

- Furthermore given q we can find a b such that,

$$\frac{t}{N} = \frac{b}{q} + \theta \pmod 1$$

with $|\theta| \leq 1/q$.

- So we get that the short sum is

$$\sum_{h\in[0,H]} e\Big(\frac{hb}{q} - \frac{h^2 a}{q} + h\theta\Big)$$

with $|\theta| \leq 1/q$.

Correcting the subconvexity

- We now apply Poisson summation:

$$\sum_{h\in[0,H]} e\Big(\frac{hb}{q} - \frac{h^2 a}{q} + h\theta\Big) \approx \frac{1}{q} \sum_{|\ell|\leq q/H} S(b-\ell, a)\mathbf{1}\Big(H\Big(\theta - \frac{\ell}{q}\Big)\Big)$$

where

$$S(a-\ell, b) = \sum_{x \pmod q} e\Big(\frac{x(a-\ell)}{q} + \frac{x^2 b}{q}\Big) \ll \sqrt{q}$$

- In particular bounding the right hand side trivially we get

$$\sum_{h\in[0,H]} e\Big(\frac{hb}{q} - \frac{h^2 a}{q} + h\theta\Big) \ll \frac{\sqrt{q}}{H} \leq H^{1-\delta/2}$$

since $q \leq H^{2-\delta}$.

Correcting the subconvexity bound

- This gives the first claim: that if

$$\frac{t}{2N^2} = \frac{a}{q} + O\Big(\frac{1}{H^2}\Big)$$

  with $H^\delta \leq q \leq H^{2-\delta}$ then the short sum over $[N, N+H]$ is bounded by $H^{1-\delta/2}$.

- It remains to show that the number of intervals $[N, N+H]$ with

$$\frac{t}{2N^2} = \frac{a}{q} + O\Big(\frac{1}{H^{2-\delta}}\Big)$$

  and $q \leq H^\delta$ is small.

Correcting the subconvexity bound

- We wish to show,

$$\sum_{kH\sim\sqrt{T}} \mathbf{1}\Big(\exists q \leq H^{\delta} : \Big\|\frac{t}{2(kH)^2} - \frac{a}{q}\Big\| \leq \frac{1}{H^{2-\delta}}\Big) \ll \frac{\sqrt{T}}{H} \cdot H^{-\eta}$$

  for some $\eta > 0$.

- We can drop the $\exists$ by using the union bound. Bounding the above by

$$\sum_{\substack{q\leq H^{\delta}\\(a,q)=1}} \sum_{kH\sim\sqrt{T}} \mathbf{1}\Big(\Big\|\frac{t}{2(kH)^2} - \frac{a}{q}\Big\| \leq \frac{1}{H^{2-\delta}}\Big)$$

- In particular it's enough to show

$$\sum_{kH\sim\sqrt{T}} \mathbf{1}\Big(\Big\|\frac{t}{2(kH)^2} - \frac{a}{q}\Big\| \leq \frac{1}{H^{2-\delta}}\Big) \ll \frac{\sqrt{T}}{H} H^{-2\delta-\eta}$$

  for some $\eta > 0$.

Correcting the subconvexity bound

- As usual we expand into a trigonometric series

$$\mathbf{1}\Big(\Big\|\frac{t}{2(kH)^2} - \frac{a}{q}\Big\| \leq \frac{1}{H^{2-\delta}}\Big) \approx \frac{1}{H^{2-\delta}} + \sum_{0<|\ell|\leq H^{2-\delta}} e\Big(\frac{\ell t}{2(kH)^2}\Big)$$

- The main term is

$$\frac{\sqrt{T}}{H^{3-\delta}} \ll \frac{\sqrt{T}}{H} H^{-2\delta-\eta}$$

for some $\eta > 0$, provided that δ is sufficiently small.

- The error term is

$$\frac{1}{H^{2-\delta}} \sum_{|\ell|\leq H^{2-\delta}} \sum_{k\sim\sqrt{T}/H} e\Big(\frac{\ell t}{2k^2 H^2}\Big)$$

Correcting the subconvexity bound

- We apply Poisson summation in k. The new length is

$$\varphi'\Big(\frac{\sqrt{T}}{H}\Big) \approx \frac{H^{2-\delta}T}{(\sqrt{T}/H)^3 H^2} \approx \frac{H^{3-\delta}}{\sqrt{T}}$$

- If H is sufficiently small power of T then this is < 1. This means that only the central term survives and therefore the behavior of the sum is exactly the sum as the integral

$$\int_{x\sim\sqrt{T}/H} e\Big(\frac{\ell t}{2x^2H^2}\Big)dx \ll \frac{\sqrt{T}}{H^{3-\delta}}$$

by the first derivative test.

- This is exactly the same bound as we obtained from the main term.

Correcting the subconvexity bound

- To summarize: we split the sum
$$\sum_{n \sim \sqrt{T}} n^{it}$$
into $\sqrt{T}/H$ intervals of length H.

- If on the interval $[N, N+H]$ we have,
$$\frac{t}{2N^2} = \frac{a}{q} + O\Big(\frac{1}{H^2}\Big)$$
for some $H^{\delta} \leq H^{2-\delta}$, then we can bound the contribution of this interval by $H^{1-\delta/2}$.

- The number of remaining intervals is (provided that H is choosen a small power),
$$\ll \frac{\sqrt{T}}{H^{3-\delta}} H^{2\delta}$$
and this is less than $\sqrt{T}H^{-1-\eta}$ for some $\eta > 0$ provided that δ is sufficiently small.

Correcting the subconvexity

- These two together give us a subconvex bound for the Riemann zeta-function.
- If you go through the proof carefully you see that we also get an algorithm for computing the Riemann zeta function in time $O(T^{1/2-\delta})$ for some $\delta > 0$.

Consequences of bounds for $\zeta(s)$

- We established a subconvex bound

$$|\zeta(\tfrac{1}{2} + it)| \ll (1 + |t|)^{1/4-\delta}$$

  for some $\delta > 0$.

- In reality we expect

$$|\zeta(\tfrac{1}{2} + it)| \ll_\varepsilon (1 + |t|)^{\varepsilon}$$

  for any given $\varepsilon > 0$.

- This is called the Lindelof Hypothesis and is the strongest bound that one can hope for (at the scale of power-savings).

Lindelof Hypothesis

- It is easy to motivate the Lindelof Hypothesis: We can write

$$\zeta(\tfrac{1}{2} + it) \approx \sum_{n \leq T} \frac{1}{n^{1/2+it}}$$

- Since the frequency p^{it} are uncorrelated it is reasonable to think of n^{it} as a random multiplicative function X_n with X_p uniformly distributed on the unit circle.
- We have,

$$\sum_{n \leq T} \frac{X_n}{\sqrt{n}} \ll T^{\varepsilon}$$

with very high probability

Lindelof Hypothesis

- Alternatively we can compute the so-called second moment,

$$\int_{|t|\leq T} |\zeta(\tfrac{1}{2}+it)|^2 \approx \int_{|t|\leq T} \Big| \sum_{n\leq T} \frac{1}{n^{1/2+it}} \Big|^2 dt \sim T \sum_{n\leq T} \frac{1}{n}$$

- Which shows that for typical t

$$|\zeta(\tfrac{1}{2}+it)| \ll \log T$$

- This is somewhat misleading though, because this bound is not true for all $t \in [T, 2T]$. It is too optimistic.

Consequences of the Lindelof Hypothesis

- If the Lindelof Hypothesis is true then we can approximate the ζ function by short Dirichlet polynomials.
- We have, for $\sigma > \frac{1}{2}$ and W a smooth compactly supported function with $W(0) = 1$,

$$\sum_{n \leq X} \frac{1}{n^{\sigma+it}} W\Big(\frac{n}{X}\Big) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(\sigma + it + w) \widetilde{W}(w) X^w dw$$

- Shifting contours to $\Re s = \frac{1}{2} - \sigma$ we get

$$\zeta(s) + \frac{1}{2\pi i} \int_{\frac{1}{2}-\sigma-i\infty}^{\frac{1}{2}-\sigma+i\infty} \zeta(\sigma + it + w) \widetilde{W}(w) X^w dw$$

- On the Lindelof Hypothesis we can bound the integral by $\ll_\varepsilon T^\varepsilon X^{1/2-\sigma}$ which is negligible as soon as $X > T^{10\varepsilon}$, say.

Consequence of the Lindelof Hypothesis

Theorem

Assume the Lindelof Hypothesis. Let $\varepsilon > 0$ be given. Then, for $\sigma > \frac{1}{2}$ and $t \in [T, 2T]$,

$$\zeta(\sigma + it) = \sum_{n \leq T^{\varepsilon}} \frac{1}{n^{1/2+it}} + O(T^{-(\sigma - 1/2)\varepsilon}).$$

- In other words on the Lindelof Hypothesis we can compute the Riemann zeta-function off the critical line in time $O(T^{\varepsilon})$.

Consequences of the Lindelof Hypothesis

- Another essentially similar consequence of the Lindelof Hypothesis is the bound

$$\sum_{n \leq N} n^{it} \ll_{\varepsilon} |t|^{\varepsilon} \sqrt{N}$$

- We will later see what these bounds have to say about the zeros of the Riemann ζ-function.
- In particular we will discuss so called zero-density theorems: theorems that establish bounds for the number of points $\beta + i\gamma$ with

$$\zeta(\beta + i\gamma) = 0 \ , \ \beta > \sigma \ , \ |\gamma| \leq T.$$

- The trivial bound is $\ll T \log T$ since this is the total number of zeros with $|\gamma| \leq T$.

Mollifiers

- A mollifier is a finite Dirichlet polynomial with the property that "most of the time"

$$\zeta(s)M(s) \approx 1$$

- Since

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}$$

where μ is the Mobius function, it is natural to expect that

$$\sum_{n \leq N} \frac{\mu(n)}{n^s}$$

should be a mollifier.

- By Mobius inversion

$$\zeta(s)M(s) = 1 + \sum_{n > N} \frac{a(n)}{n^s}$$

where $|a(n)| \leq d(n) \ll_\varepsilon n^\varepsilon$.

Mollifiers

- We are interested in bounding

$$N(\sigma; T) := \Big\{\beta + i\gamma : \zeta(\beta + i\gamma) = 0, \beta > \sigma, |\gamma| \leq T\Big\}.$$

- Trivially

$$\begin{aligned} N(\sigma; T) &\leq \frac{1}{\varepsilon} \sum_{\substack{\beta > \sigma - \varepsilon \\ |\gamma| \leq T \\ \zeta(\beta + i\gamma)}} (\beta - \tfrac{1}{2}) \\ &\leq \frac{1}{\varepsilon} \sum_{\substack{\beta > \sigma - \varepsilon \\ |\gamma| \leq T \\ (M\zeta)(\beta + i\gamma) = 0}} (\beta - \tfrac{1}{2}) \end{aligned}$$

- By Littewood's formula (an analogue of Jensen formula for rectangles) the above is

$$\leq \frac{1}{\varepsilon} \int_{|t| \leq T} \log |(\zeta M)(\sigma - \varepsilon + it)| dt$$

Mollifiers

- By Jensen's inequality

$$\int_{|t|\leq T} \log|(M\zeta)(\sigma+it)|dt \leq T\log\Big(\frac{1}{2T}\int_{|t|\leq T}|(\zeta M)(\sigma+it)|^2 dt\Big)$$

- One typically simply computes the above second moment, and this is the "classical" way of obtaining a zero-density estimate.
- For instance if we M is a mollifier of length X, then we expect

$$\frac{1}{2T}\int_{|t|\leq T}|(\zeta M)(\sigma+it)|^2 dt = 1+O\Big(\sum_{n>X}\frac{1}{n^{2\sigma}}\Big) = 1+O(X^{-(2\sigma-1)})$$

Mollifiers

- And this would lead to a zero density theorem of the form,

$$N(\sigma, T) \ll TX^{-(2\sigma-1)}$$

- In practice it is fairly easy to compute such expressions with $X = T^{1/2}$ and this leads to a standard zero-density bound of the form

$$N(\sigma, T) \ll \frac{1}{\varepsilon} \cdot TX^{-(2\sigma-1+2\varepsilon)}$$

- A reasonable choise of ε is $1/\log T$ and this would then give

$$\ll T^{1-(\sigma-\frac{1}{2})} \log T$$