Very weak bounds for $\zeta(s)$

- Trivially we have, for $\sigma > 1 + \varepsilon$,

$$\zeta(\sigma + it) \ll_\varepsilon 1$$

- By the functional equation we also have for $\sigma < -\varepsilon$,

$$|\zeta(\sigma + it)| \ll (1 + |t|)^{\frac{1}{2} - \sigma}$$

- We will now use convexity in complex analysis to get a preliminary (very weak) bound
- This bound is just needed to ensure the convergence of the various integrals that we will consider. We will subsequently obtain approximations from which stronger bounds will follow.

Very weak bounds for $\zeta(s)$

- Let $0 < \Re s < 1$. By Cauchy's theorem,

$$\zeta(s) = \frac{1}{2\pi i} \int_{\sigma^+ + \varepsilon - i\infty}^{\sigma^+ + \varepsilon + i\infty} \zeta(s+w) \frac{e^{w^2} dw}{w} - \frac{1}{2\pi i} \int_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} \zeta(s+w) \frac{e^{w^2} dw}{w}$$

  We pick σ^+ and σ^- so that $\sigma^+ + \Re s = 1 + \varepsilon$ and $\sigma^- + \Re s = -\varepsilon$.

- Bounding the integrals we get

$$|\zeta(s)| \ll_\varepsilon (1 + |t|)^{1/2+\varepsilon}$$

  in the region $0 < \Re s < 1$.

- Incidentally this proof essentially shows that,

$$f(\sigma) = \sup_{|t| \le T} |\zeta(\sigma + it)|$$

  grows as σ decreases.

Convexity

- The observation that

$$f(\sigma) = \sup_{|t| \leq T} |\zeta(\sigma + it)|$$

  is basically growing when σ is decreasing is important.
- It means that to understand the growth of the Riemann ζ function we can focus on $\sigma = \frac{1}{2}$.
- This is the hardest case.

Approximating ζ in the critical strip

- We now derive two useful approximations for the Riemann ζ function inside the critical strip.
- Pick W a smooth function with $W(0) = 1$ and $W(x) = 0$ for $x > 10$, say. Let $\varepsilon > 0$ be given.
- We then have, for $t \in [T, 2T]$,

$$\sum_{n \geq 1} \frac{1}{n^{\sigma+it}} W\Big(\frac{n}{X}\Big) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s+w) X^w \widetilde{W}(w) dw$$

- We now shift contours to $\Re w = -A$. This gives

$$W(0)\zeta(s) + \frac{1}{2\pi i} \int_{-A-i\infty}^{-A+i\infty} \zeta(s+w) X^w \widetilde{W}(w) dw$$

- The second term is small if $X > T^{1+\varepsilon}$ for any given ε. In fact it is then bounded by $O(T^{-\varepsilon A})$.

First approximation

- So this gives us a first approximation to the Riemann ζ function inside the critical strip

Theorem

Let $\varepsilon, A > 0$ be given. Let W be a smooth function with $W(0) = 1$ and $W(x) = 0$ for sufficiently large x. For any $0 < \sigma < 1$ and $t \in [T, 2T]$,

$$\zeta(\sigma + it) = \sum_{n \geq 1} \frac{1}{n^{\sigma + it}} W\Big(\frac{n}{T^{1+\varepsilon}}\Big) + O_A(T^{-A})$$

- This formula gives us an algorithm for computing the Riemann zeta function in time basically $O(1 + |t|)$.

Second approximation

- One can derive a better approximation by using the functional equation for $\zeta(s)$.
- However since the functional equation is equivalent to Poisson summation we could instead simply apply Poisson summation on the previous approximation. This is what we are going to do.
- To do this note that one can construct a smooth compactly supported function V such that,

$$\sum_{k\geq 1} V\left(\frac{n}{2^k}\right) = 1$$

for every integer $n \geq 1$. This is called a partition of unity.

Second approximation

- We now apply the partition of unity, so that

$$\zeta(1/2+it) = \Big(\sum_{2^k < T^{1/2}} + \sum_{2^k > T^{1/2}} \Big) \sum_{n \geq 1} \frac{1}{n^{1/2+it}} W\Big(\frac{n}{T^{1+\varepsilon}}\Big) V\Big(\frac{n}{2^k}\Big)$$

- We leave the terms with $2^k < T^{1/2}$ as is.
- On the terms $2^k > T^{1/2}$ we apply Poisson summation.
- To make things simple we can prented that $W \equiv 1$ and that 2^k runs only up to T.

Poisson summation

- Poisson summation gives

$$\sum_n \frac{1}{n^{1/2+it}} V\Big(\frac{n}{2^k}\Big) = \sum_k \int_{\mathbb{R}} \frac{1}{x^{1/2+it}} V\Big(\frac{x}{2^k}\Big) e(-kx) dx$$

  where $e(x) := e^{2\pi i x}$.

- To understand the integral we need to apply stationary phase.
- The idea is to carefully consider the behavior of the phase

$$x^{-it} e(-kx) = e(-it \log x - kx) = e(\varphi(x))$$

- Whenever x_0 is a point such that $\varphi'(x_0) \neq 0$ the phase will oscilate in the neighborhood of x_0 and so we expect no contribution to the integral from that neighborhood.

Stationary phase

- Recall that
$$e(\varphi(x)) = e(-it\log x - kx)$$
- When $\varphi'(x_0) = 0$ we have,
$$e(\varphi(x)) = e\Big(\varphi(x_0) + \frac{1}{2}(x - x_0)^2\varphi''(x_0) + \dots\Big)$$
and so we expect a main term contribution in a $1/\sqrt{\varphi''(x_0)}$ neighborhood of x_0.
- Thus we expect
$$\int_{\mathbb{R}} \frac{1}{\sqrt{x}} V\Big(\frac{x}{2^k}\Big) x^{-it} e(-kx)dx = c \sum_{\varphi'(x_0)=0} \frac{e(\varphi(x_0))}{|\sqrt{\varphi''(x_0)}|} \frac{1}{\sqrt{x_0}} V\Big(\frac{x_0}{2^k}\Big)$$
with some constant c that we don't care about.

Stationary phase

- Doing this work we find

$$\sum_n \frac{1}{n^{1/2+it}} V\Big(\frac{n}{2^k}\Big) \approx e^{-it\log t} \sum_n \frac{1}{n^{1/2-it}} V\Big(\frac{n}{T/2^k}\Big)$$

- An important sanity check is that the leading harmonic oscillates at the same speed
- Note however that there are fewer terms in the second sum since $2^k > T^{1/2}$.

Approximate functional equation

- To summarize: We wrote

$$\zeta(\tfrac{1}{2}+it) = \Big(\sum_{2^k < T^{1/2}} + \sum_{2^k > T^{1/2}}\Big) \sum_{n \geq 1} \frac{1}{n^{1/2+it}} W\Big(\frac{n}{T^{1+\varepsilon}}\Big) V\Big(\frac{n}{2^k}\Big)$$

- Applying Poisson summation we then get

$$\sum_{2^k < T^{1/2}} \Big(\sum_{n} \frac{1}{n^{1/2+it}} V\Big(\frac{n}{2^k}\Big)\Big) + \sum_{2^k > T^{1/2}} \Big(e^{-it\log t} \sum_{n} \frac{1}{n^{1/2-it}} V\Big(\frac{n}{T/2^k}\Big)\Big)$$

- And therefore

$$\zeta(\tfrac{1}{2}+it) = \sum_{n < T^{1/2}} \frac{1}{n^{1/2+it}} + e^{-it\log t} \sum_{n < T^{1/2}} \frac{1}{n^{1/2-it}}$$

- This is known as the approximate functional equation.

Approximate functional equation

- The approximate functional equation gives an algorithm for computing the Riemann zeta function in time $O(\sqrt{T})$. Using the trivial bound also gives us the bound

$$|\zeta(\tfrac{1}{2} + it)| \ll (1 + |t|)^{1/4}$$

  which is known as the convexity bound.
- In reality we can further group the terms and obtain an algorithm that works in time $O(T^{1/3})$.
- This grouping of terms actually gives a bound of the form $|\zeta(\frac{1}{2} + it)| \ll (1 + |t|)^{1/6}$ which is a subconvex bound.

Subconvex bounds

- I will now explain how this more efficient representation can be obtained.
- This will be a very brief introduction to the Bombieri-Iwaniec and Jutila method for bounding exponential sums.
- The optimal bound that one could hope for is $|\zeta(\frac{1}{2}+it)| \ll_\varepsilon (1+|t|)^\varepsilon$. This would have important consequences for prime numbers. We will explore these in the next lecture.
- Obtaining bounds that beat the convexity bound is a big industry in analytic number theory. Often these bounds have arithmetic significance.

Subconvexity

- Our goal is therefore to obtain a bound for
$$\sum_{n \leq \sqrt{T}} \frac{1}{n^{1/2+it}}$$
that beats the trivial bound $T^{1/4}$
- To make things simple it's enough to focus on the problem of bounding non-trivially
$$\sum_{n \sim \sqrt{T}} n^{it}$$
- The idea is simple: we split this sum into short sums of length T^{δ} and we hope to obtain a non-trivial saving on each of these sums.

Subconvexity

- We are therefore looking at a sum of the form

$$\sum_{n\sim\sqrt{T}} n^{it} \approx \sum_k \sum_n n^{it} W\Big(\frac{n-N_k}{H}\Big)$$

with $N_k = kH \asymp \sqrt{T}$ and $H = T^{\delta}$. We will write $N = N_k$.

- Notice that for $n = N + h$ with $0 \le h \le H$,

$$\begin{aligned} n^{it} &= N^{it} \cdot \Big(1 + \frac{h}{N}\Big)^{it} = N^{it} e\Big(\frac{ht}{N} + \dots\Big) \\ &\approx N^{it} e\Big(\frac{ht}{N}\Big) \end{aligned}$$

where $\dots$ is negligible because $H^2\sqrt{T}/N^2 = T^{2\delta-1/2}$ (WARNING: this is wrong; the second term is not negligible).

Error

- WARNING: One in fact needs two terms in the Taylor series.
- Just using one term is enough to understand the gist of the method , but is a gross oversimplification.
- This is rectified in the next lecture

Subconvexity

- We now do write
$$\frac{t}{N} = \frac{a}{q} + \theta_N$$
with $|\theta_N| \leq 1/(qH)$ and $q \leq H$.
- Thus,
$$\sum_n n^{-it} W\Big(\frac{n-N}{H}\Big) \approx N^{it} \sum_h e\Big(\frac{ha}{q}\Big) e(h\theta_N) W\Big(\frac{h}{H}\Big)$$
- We can use Poisson summation to understand the second sum (or just sum the geometric series). The upshot is that there is at most one term in the sum.
- Roughly speaking the sum is either bounded (in which case we win) or it is of size H, but then $q = 1$ and
$$\Big\| \frac{t}{N} \Big\| \leq \frac{1}{H}$$

Subconvexity

- Therefore we obtain the following bound

$$|\zeta(\tfrac{1}{2}+it)| \ll T^{1/4}/H + H\#\mathcal{P}/T^{1/4} + H^2/T^{1/4}$$

  where $\mathcal{P}$ is a set of H well-spaced points $N_1, N_2, \ldots$ at which

$$\left\| \frac{t}{N_i} \right\| \le \frac{1}{H}.$$

- The term $T^{1/4}/H$ comes from the "good" short intervals on which we can have cancellations
- The term $H\#\mathcal{P}$ is the contribution of "bad" short intervals on which we have

$$\left\| \frac{t}{N_i} \right\| \le \frac{1}{H}$$

- And finally $H^2/T^{1/4}$ is the contribution from the rational approximation of the phase

Subconvexity

- So to conclude it simply remains to show that if N_i is a set of well-spaced points, say $N_i = kH$ then the number of $k \sim T^{1/2}/H$ at which

$$\left\| \frac{t}{kH} \right\| \leq \frac{1}{H}$$

is small.

- This is the same as computing

$$\sum_{k \sim T^{1/2}/H} \mathbf{1}\Big(\left\| \frac{t}{kH} \right\| \leq \frac{1}{H} \Big)$$

- We can expand the indicator function in Fourier series. Roughly speaking

$$\mathbf{1}\Big(\left\| \frac{t}{kH} \right\| \leq \frac{1}{H} \Big) \approx \frac{1}{H} + \frac{1}{H} \sum_{0 < |\ell| \leq H} e\Big(\frac{\ell t}{kH} \Big)$$

Subconvexity

- Therefore the number of $k \sim T^{1/2}/H$ is

$$\frac{T^{1/2}}{H^2} + \frac{1}{H} \sum_{0<|\ell|<H} \sum_{k \sim T^{1/2}/H} e\Big(\frac{\ell t}{kH}\Big)$$

- The sum over k can be estimated by simply applying Poisson summation. Here is a simple ansatz:

$$\sum_{k \sim K} e(\varphi(k)) = (\star) \sum_{k \sim \varphi'(K)} e(\varphi^{\star}(k))$$

and the normalization $(\star)$ is choosen so as to preserve square-root cancellation: namely $(\star) = \sqrt{K}/\sqrt{\varphi'(K)}$.

Subconvexity

- From this ansatz we see that

$$\sum_{k \sim T^{1/2}/H} e\Big(\frac{\ell t}{kH}\Big) = \frac{T^{1/4}}{H} \sum_{k \sim H} e(\dots) \ll T^{1/4}.$$

since the phase

$$\varphi'\Big(\frac{T^{1/2}}{H}\Big) \asymp \frac{H\sqrt{T}}{\sqrt{T}} = H$$

- We conclude that,

$$\#\mathcal{P} \ll \frac{T^{1/2}}{H^2} + T^{1/4}$$

Final bound

- We therefore have obtained

$$|\zeta(\tfrac{1}{2}+it)| \ll \frac{T^{1/4}}{H} + H + \frac{H^2}{T^{1/4}}$$

- Balancing these two expressions the optimal choice is

$$H = T^{1/12}$$

  and this gives

$$|\zeta(\tfrac{1}{2}+it)| \ll T^{1/6}.$$

- This bound is known as Weyl subconvexity. It is an recurrent barrier in many problems.