# Type I/II sums and sieves

Recap: – Type I not sufficient to show $\exists$ primes in $\mathcal{A}$

– If $\mathcal{A}$ has Type I $[0,\alpha]$, Type II $[\beta,\gamma]$ with $\alpha+(\gamma-\beta)\geq 1$

then asymptotic for $\#\{p\in\mathcal{A}\}$.

– If $\mathcal{A}$ has Type I $[0,\alpha]$, Type II $[\beta,\gamma]$ with $\alpha+(\gamma-\beta)=1-\delta$

then <u>can't</u> necessarily get asymptotic formula for $\#\{p\in\mathcal{A}\}$.

<u>Can</u> get asymptotic up to a factor $(1+O(\delta))$.

$\therefore$ lower bounds $\#\{p\in\mathcal{A}\}\gg \frac{\#\mathcal{A}}{\log x}$.

$*$ Key point: Can often get a lower bound with only numerically small Type II intervals $*$.

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Revisit example from last time:

Imagine that $\mathcal{A}\subseteq[x,2x]$ which satisfies Type I $[0,\frac{2}{3}-\delta]$, Type II $[\frac{1}{6}-\delta,\frac{1}{3}-\delta]$.

$$S(\mathcal{C}_d,z):=S(\mathcal{A}_d,z)-\lambda S(\mathcal{B}_d,z).$$

Want to estimate $S(\mathcal{A},2x^{\frac{1}{2}})$ $\therefore$ we want to estimate $S(\mathcal{C},2x^{\frac{1}{2}})$.

- By our Type I + Type II + Proposition, can estimate $S(\mathcal{C}_d,z)$ when $z\leq x^{\gamma-\beta}=x^{\frac{1}{6}}$, $d\leq x^{\alpha}=x^{\frac{2}{3}-\delta}$.
- By our Type II estimate, we can estimate $S(\mathcal{C}_d,z)$ whenever $d_1|d$ with $d_1\in[x^{\beta},x^{\gamma}]=[x^{\frac{1}{6}-\delta},x^{\frac{1}{3}-\delta}]$.
- By non-negativity $S(\mathcal{C}_d,z)=S(\mathcal{A}_d,z)-\lambda S(\mathcal{B}_d,z)\geq-\lambda S(\mathcal{B}_d,z)$.

So,

$$S(\mathcal{C},2x^{\frac{1}{2}})=S(\mathcal{C},x^{\frac{1}{6}})-\sum_{x^{\frac{1}{6}}\leq p\leq 2x^{\frac{1}{2}}}S(\mathcal{C}_p,p)$$

$\leftarrow$ Can estimate using Type II if $p\in[x^{\frac{1}{6}-\delta},x^{\frac{1}{3}-\delta}]$.

$$=o\left(\frac{\#\mathcal{A}}{\log x}\right)-\sum_{x^{\frac{1}{3}-\delta}\leq p\leq x^{\frac{1}{3}+2\delta}}S(\mathcal{C}_p,p)-\sum_{x^{\frac{1}{3}+2\delta}\leq p\leq x^{\frac{1}{2}}}S\left(\mathcal{C}_p,\left(\frac{2x}{p}\right)^{\frac{1}{2}}\right)$$

Negligible by our proposition since $\left(\frac{2x}{p}\right)^{\frac12} \le x^{\frac16}$.

$$= o\left(\frac{\#A}{\log x}\right) - \sum_{x^{\frac13-\delta} \le p \le x^{\frac13+2\delta}} S(C_p, x^{\frac16}) + \sum_{\substack{x^{\frac13-\delta} \le p \le x^{\frac13+2\delta} \\ x^{\frac16} \le q \le p}} S(C_{pq}, q)$$

Negligible by proposition.

Negligible by Type II if $q \in [x^{\frac16-\delta}, x^{\frac13-\delta}]$

$$= o\left(\frac{\#A}{\log x}\right) + \sum_{x^{\frac13-\delta} \le q \le p \le x^{\frac13+2\delta}} S(C_{pq}, q)$$

$$\ge -\lambda \sum_{x^{\frac13-\delta} \le q \le p \le x^{\frac13+2\delta}} S(B_{pq}, q)$$

This counts products of 3 primes in B can get asymptotic.

$$\therefore S(A, 2x^{\frac12}) \ge \lambda S(B, 2x^{\frac12}) - \lambda \sum_{x^{\frac13-\delta} \le q \le p \le x^{\frac13+2\delta}} S(B_{pq}, q)$$

$$= \mathfrak{S} \frac{\#A}{\log x}\left(1 - \iint_{\frac13-\delta \le u \le v \le \frac13+2\delta} \frac{du\, dv}{uv(1-u-v)} + o(1)\right)$$

if $\delta \le \frac{1}{12}$ so this counts products of 3 primes

In fact, numerically evaluating this integral gives a good lower bound for all $\delta \le \frac{1}{12}$.

$\therefore$ get $\#\{p \in A\} \gg \frac{\#A}{\log x}$.

We 'expect' $\#\{p \in A\} \approx \mathfrak{S} \frac{\#A}{\log x} = \lambda S(B, 2x^{\frac12})$. so $\lambda = \frac{\mathfrak{S}\#A}{x}$ if $B = [x, 2x]$

Partial summation + P.N.T.

$$\sum_{x^{\frac13-\delta} \le q \le p \le x^{\frac13+2\delta}} S(B_{pq}, q) = (1+o(1))\#B \iint_{\frac13-\delta \le u \le v \le \frac13+2\delta} \frac{du\, dv}{uv(1-u-v)}$$

$$\| \qquad\qquad \| $$

$$\#\{pqr \in B : p,q,r \text{ primes},\ x^{\frac13-\delta} \le q \le p \le x^{\frac13+2\delta}\}$$

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FACT: By being careful about inclusion-exclusion can often get a lower bound with only a small amount of Type II info.

Examples: ① Harman: For $\alpha, \beta \in \mathbb{R}$ $\exists$ $\infty$ly many primes s.t.

$$\|\alpha p + \beta\| \le p^{-\frac{7}{22}}.$$

in this case Type I: $[0, \frac{15}{22}] = [0, 0.68]$

Type II: $[\frac{7}{22}, \frac{8}{22}] = [0.318\ldots, 0.363\ldots]$

width $= 0.045\ldots$

(2) Primes with restricted digits.

Type I: $[0, 0.64]$

Type II: $[0.36, 0.425]$ width $0.065$

Unfortunately in general it is messy computation process to go from explicit Type I/II info to bounds on primes, and not understood v. well theoretically.

Q: What Type I/II info is necessary/sufficient for showing $\#\{p \in A\} \gg \frac{\#A}{\log x}$ ?

Q: What are optimal upper/lower bounds given Type I/II info? What do extremal sets look like?

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Morally, Type I estimates allow you to understand $S(A_d, x^{\varepsilon})$ well

Type I + Type II " " $S(A_d, x^{\gamma - \beta})$ well

Beyond Type I/II: In our inclusion-exclusion setup, we only needed to consider coefficients $\alpha_n, \beta_m$ which looked like the indicator function of primes.

$\hookrightarrow$ We can make use of this in some contexts!

Example 1: Friedlander–Iwaniec on primes $X^2 + Y^4$: Only establish Type II estimates for sequences $\alpha_n, \beta_m$ satisfying Siegel-Walfisz type condition.

Example 2: Indicator function of primes can be written as Type I/II sums itself.

$\therefore$ Instead can look at trilinear sums

$$\sum_{mnk \in A} \alpha_m \beta_n \gamma_k \qquad \text{(triple convolutions)}$$

$$\sum_{mnk \in A} \alpha_m \beta_n \qquad \text{(triple convolutions with smooth variable)}$$

$$\sum_{mnk \in A} \alpha_m$$ (triple convolutions with 2 smooth variables)

This is v. important in results on primes in APs beyond $x^{\frac{1}{2}}$ (e.g. BFI/Zhang).

Q: Can you get good estimates for

$$\sum_{q \leq x^{\frac{1}{2}+\delta}} \left| \sum_{\substack{n_1 n_2 n_3 n_4 \equiv 1 (q) \\ n_i \in [x^{\frac{1}{4}}, 2x^{\frac{1}{4}}]}} 1 - \frac{1}{\phi(q)} \sum_{\substack{(n_1 n_2 n_3 n_4, q) = 1 \\ n_i \in [x^{\frac{1}{4}}, 2x^{\frac{1}{4}}]}} 1 \right| ?$$ (Show this is $\ll \frac{x}{(\log x)^{100}}$).

Q: What other sorts of arithmetic information can be used in this method?
Eg. $\tau(n)$, $\tau_3(n)$ = coefficients of higher degree L-functions.
Can $\sum_{n \in A} \lambda_g(n)$ be used to get more info on $\#\{p \in A\}$?

PROOF-OF-CONCEPT: Drappeau-M. : essentially getting asymptotic for $\sum_{p \leq x} Kl(1,p)$ using only Type I info.
(this is because $Kl(1,n)$ is typically smaller on $n$ with many prime factors)

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$$\sum_{nm \in A} \alpha_n \beta_m \xrightarrow{\text{Cauchy-Schwarz}} \sum_{m_1, m_2} \beta_{m_1} \overline{\beta_{m_2}} \sum_{\substack{n \\ m_1 n \in A \\ m_2 n \in A}} 1$$

Inner sum.

If you want asymptotic formula for inner sum, can only hope to get type II in an interval $[\theta, 1-2\theta]$ if $\#A = x^{1-\theta}$
(and Type I: $[0, 1-\theta]$).