Type I/II sums and sieves

Recap: Type I <u>not</u> enough to understand primes

$$\#\{p \in A\} = S(A, x^{\varepsilon}) - \sum_{x^{\varepsilon} \le p \le (2x)^{1/2}} S(A_p, p)$$

$S(A, x^{\varepsilon})$: Can be estimated using Type I info

$\sum S(A_p, p)$: Has bilinear structure.

$\therefore$ Want to investigate sums $\sum_{nm \in A} \alpha_n \beta_m$ for <u>arbitrary</u> coefficients $|\alpha_n|, |\beta_m| \le 1$.

Convenient way to keep track of everything: Compare everything to a simpler set B.

$\therefore$ We want to show $\sum_{mn \in A} \alpha_n \beta_m \approx \lambda \sum_{mn \in B} \alpha_n \beta_m$ (for some constant λ).

Typically $B = [x, 2x]$, $\lambda = \frac{\#A}{\#B} \mathfrak{S}$ (singular series correction).

(This is saying that the probability of having a particular prime factorisation in A is the product of the prob. of being in A and prob. of having the prime factorisation).

Linear algebra interpretation: Eigenvalues of matrix $M^{(A)} - \lambda M^{(B)}$ are all small

$$M^{(A)}_{ij} = \begin{cases} 1, & ij \in A \\ 0, & \text{o/w} \end{cases}, \quad M^{(B)}_{ij} = \begin{cases} 1, & ij \in B \\ 0, & \text{o/w}. \end{cases}$$

Note: In this notation, our Type I estimate is of the form

$$\sum_{nm \in A} \alpha_n \approx \lambda \sum_{nm \in B} \alpha_n \quad \text{for any sequence } |\alpha_n| \le 1 \text{ supported on } n \le x^{\theta}.$$

(i.e. $\beta_m = 1$).

PROPOSITION: Imagine our set A satisfies: For any $|\alpha_n| \le 1$, $|\beta_m| \le 1$:

Type I estimate $[0, \alpha]$: $$\sum_{\substack{n \le x^{\alpha} \\ nm \in A}} \alpha_n = \lambda \sum_{\substack{n \le x^{\alpha} \\ nm \in B}} \alpha_n + O\left(\frac{\#A}{(\log x)^{100}}\right).$$

$mn\in A$ $\qquad$ $mn\in B$

Type II estimate $[\beta,\gamma]$: $$\sum_{\substack{mn\in A\\ n\in[x^\beta,x^\gamma]}} \alpha_n \beta_m = \lambda \sum_{\substack{nm\in B\\ n\in[x^\beta,x^\gamma]}} \alpha_n\beta_m + O\left(\frac{\#A}{(\log x)^{100}}\right).$$

For fixed reals α,β,γ with $\alpha \geq \beta$.

Then $$S(A, x^{\gamma-\beta}) = \lambda S(B, x^{\gamma-\beta}) + o\left(\frac{\#A}{(\log x)}\right).$$

Proof: Repeated applications of Buchstab's identity.

$$S(A,x^{\gamma-\beta}) = S(A,x^{\varepsilon}) - \sum_{x^\varepsilon\leq p\leq x^{\gamma-\beta}} S(A_p,p)$$

If $p\notin[x^\beta,x^\gamma]$ then keep on applying Buchstab.

Good by Type I estimates

If $p\in[x^\beta,x^\gamma]$ then good by Type II

To keep track of everything it is convenient to write
$$S(C_d,z) := S(A_d,z) - \lambda S(B_d,z).$$

$$T_n := \sum_{\substack{x^\varepsilon\leq p_n\leq\cdots\leq p_1\leq x^{\gamma-\beta}\\ p_1\cdots p_n\leq x^\beta}} S(C_{p_1\cdots p_n}, p_n)$$

$$U_n := \sum_{\substack{x^\varepsilon\leq p_n\leq\cdots\leq p_1\leq x^{\gamma-\beta}\\ p_1\cdots p_n\leq x^\beta}} S(C_{p_1\cdots p_n}, x^\varepsilon)$$ ← Can evaluate using Type I estimates + Linear sieve

$$V_n := \sum_{\substack{x^\varepsilon\leq p_n\leq\cdots\leq p_1\leq x^{\gamma-\beta}\\ p_1\cdots p_n\geq x^\beta\\ p_1\cdots p_{n-1}\leq x^\beta}} S(C_{p_1\cdots p_n}, p_n).$$ ← Can evaluate using Type II estimates.

Buchstab's Identity ⇒ $$T_n = U_n - T_{n+1} - V_{n+1}$$
$$= \sum_{j=n}^{\infty} U_j (1)^{j-n} + \sum_{j=n+1}^{\infty} V_j (-1)^{j-n}.$$

$\therefore$

$$S(C,x^{\gamma-\beta}) = S(C,x^\varepsilon) - \sum_{x^\varepsilon\leq p\leq x^{\gamma-\beta}} S(C_p,p)$$

$$\therefore \left| S(C, x^{\gamma-\beta}) \right| = o\left(\frac{\#A}{\log x}\right). \quad \overset{\shortparallel}{T_1}$$

PROPOSITION (Vaughan's Identity).

Imagine A satisfies a Type I estimate in $[0, \alpha]$

Type II estimate for $[\beta, \gamma]$.

If $(\gamma - \beta) + \alpha \geq 1$ then

$$\#\{p \in A\} = \lambda \#\{p \in B\} + o\left(\frac{\#A}{\log x}\right).$$

PROOF: Note: That $S(A_{p_1 \cdots p_n}, p_n)$ counts products $p_1 \cdots p_n q \in A$

with $P^-(q) \geq p_n$, and $q \sim \frac{x}{p_1 \cdots p_n}$

$\therefore$ If $p_n \geq q^{1/2} = \left(\frac{2x}{p_1 \cdots p_n}\right)^{1/2}$ then this forces q to be prime.

$$\therefore S(A_{p_1 \cdots p_n}, p_n) = S\left(A_{p_1 \cdots p_n}, \left(\frac{2x}{p_1 \cdots p_n}\right)^{1/2}\right).$$

In particular, $S(A_p, p) = S\left(A_p, \min\left(p, \left(\frac{2x}{p}\right)^{1/2}\right)\right)$, (for $p \leq x^{1/2}$).

$\#\{p \in A\} = S(A, (2x)^{1/2})$ $\therefore$ we want to show $S(C, (2x)^{1/2}) = o\left(\frac{\#A}{\log x}\right)$.

$$S(C, (2x)^{1/2}) = S(\overset{d(1)}{C}, x^{\gamma-\beta}) - \underbrace{\sum_{x^{\gamma-\beta} \leq p \leq (2x)^{1/2}} S(C_p, p)}_{= T_1}.$$

Let $$T_n = \sum_{\substack{x^{\gamma-\beta} \leq p_n \leq \cdots \leq p_1 \leq (2x)^{1/2} \\ p_1 \cdots p_{j-1} p_j^2 \leq x \ \forall j}} S(C_{p_1 \cdots p_n}, p_n)$$

Buchstab $\Rightarrow T_n = U_n - T_{n+1}$

$$U_n = \sum_{\substack{x^{\gamma-\beta} \leq p_n \leq \cdots \leq p_1 \leq (2x)^{1/2} \\ p_1 \cdots p_{j-1} p_j^2 \leq x \ \forall j}} S(C_{p_1 \cdots p_n}, x^{\gamma-\beta})$$

since

$$S(C_{p_1 \cdots p_n}, p_n) = S\left(C_{p_1 \cdots p_n}, \min\left(p_n, \left(\frac{x}{p_1 \cdots p_n}\right)^{1/2}\right)\right)$$

$$\therefore T_1 = \sum_{j=1}^{\infty} (-1)^{j-1} U_j.$$

$\therefore$ suffices to show that $\#U_j = o\left(\frac{\#A}{\log x}\right)$ for all j.

But this follows from the same argument as previous proposition!

In U_n, $p_1 \cdots p_{n-1} p_n^2 \leq x$ and $p_n \geq x^{\gamma-\beta}$

$$\Rightarrow p_1 \cdots p_n \leq x^{1-(\gamma-\beta)} \leq x^{\alpha} \quad \text{since } \alpha + (\gamma-\beta) \geq 1.$$

Slight generalisation of previous proposition shows that

$$\sum_{\substack{p_n, \dots, p_1 \\ p_1 \cdots p_n \leq x^{\alpha} \\ \text{(other linear inequalities)}}} S(\mathcal{C}_{p_1 \cdots p_n}, x^{\gamma-\beta}) = o\left(\frac{\#A}{\log x}\right).$$

Remarks:
- If $\alpha = 1-\varepsilon$ then you only need a Type II estimate of width ε to break parity and get asymptotic formula for primes.
- This inclusion-exclusion approach naturally generalises to give non-trivial lower bounds even if $\alpha + (\beta-\gamma) < 1$ provided it is big enough.

Example: Type I: $[0, \frac{2}{3} - \delta]$ so $\alpha + (\gamma - \beta) = 1 - \delta$
Type II: $[\frac{1}{6} - \delta, \frac{1}{3} - \delta]$

$$S(\mathcal{C}, 2x^{\frac{1}{2}}) = \underbrace{S(\mathcal{C}, x^{\frac{1}{6}})}_{o(1)} - \underbrace{\sum_{x^{\frac{1}{6}} \leq p \leq x^{\frac{1}{3}-\delta}} S(\mathcal{C}_p, p)}_{o(1)} - \sum_{x^{\frac{1}{3}-\delta} \leq p \leq 2x^{\frac{1}{2}}} S(\mathcal{C}_p, p)$$

$$= \sum_{\substack{x^{\frac{1}{3}-\delta} \leq p \leq 2x^{\frac{1}{2}} \\ x^{\frac{1}{6}} \leq q \leq \min(p, (\frac{x}{p})^{\frac{1}{2}})}} S(\mathcal{C}_{pq}, q) = \sum_{\substack{x^{\frac{1}{3}-\delta} \leq p \leq 2x^{\frac{1}{3}+2\delta} \\ x^{\frac{1}{3}-\delta} \leq q \leq \left\{ \begin{smallmatrix} p \\ (\frac{x}{p})^{1/2} \end{smallmatrix} \right.}} S(\mathcal{C}_{pq}, q) = \text{products of 3 primes all close to } x^{\frac{1}{3}} \ (\in [x^{\frac{1}{3}-2\delta}, x^{\frac{1}{3}+2\delta}])$$

Since products of 3 primes are in short intervals on a log-scale, we see

$$S(\mathcal{C}, x^{\frac{1}{2}}) \ll \frac{\delta \#A}{\log x}, \text{ so if } \delta \text{ is small enough}$$

$$S(A, 2x^{\frac{1}{2}}) \gg \frac{\#A}{\log x}.$$

But FACT: $\exists$ sets A which satisfy this Type I / II estimates but have different asymptotics for the number of primes,