Type I/II sums and sieves

Recap: Type I not enough to understand primes

\[ \#\text{spe}\, A? = S(A, x) - \sum_{x < p < (x)^{1/2}} S(A, p) \]

\[ \uparrow \]

Can be estimated using Type I \[ \uparrow \]

Has bilinear structure.

\[ \therefore \text{Want to investigate sums } \sum_{n \in A} \beta_n \text{ for arbitrary coefficients } |\alpha_n|, |\beta_n| \leq 1. \]

Convenient way to keep track of everything: Compare everything to a simpler set B.

\[ \therefore \text{We want to show } \sum_{n \in A} \beta_n = a \sum_{n \in B} \beta_n \text{ (for some constant } a). \]

Typically \[ B = \mathbb{E}_x [x], \quad a = \frac{\#A}{\#B} G \]

(singular value correction).

(This is saying that the probability of having a particular prime factorization in A is the product of the prob. of being in A and prob. of having the prime factorization).

Linear algebra interpretation: Eigenvalues of matrices \[ M^{(A)} - a M^{(B)} \]

are all small

\[ M^{(A)}_{ij} = \mathbb{E}_i, \quad i \in A, \quad a/\omega, \quad M^{(B)}_{ij} = \mathbb{E}_j, \quad i \in B, \quad a/\omega. \]

Note: In this notation, our Type I estimate is of the form

\[ \sum_{n \in A} \alpha_n = a \sum_{n \in B} \alpha_n \]

for any sequence \( |\alpha_n| \leq 1, \) supported on \( n < x^e. \)

(i.e. \( \beta_n = 1 \)).

Proposition: Imagine our set A satisfies. For any \( |\alpha_n| \leq 1, \) \( |\beta_n| \leq 1:\)

Type I estimate \([0, a]\):

\[ \sum_{n \leq x^e} \alpha_n = a \sum_{n \leq x^e} \alpha_n + O\left( \frac{\#A}{(\log x)^{100}} \right). \]
Type II estimates $E, B$: 

\[ \sum \alpha \beta m = \sum \alpha \beta m + o\left( \frac{#A}{(\log x)^C} \right). \]

For fixed reals $\alpha, \beta, \gamma$ with $\alpha, \beta, \gamma > 0$.

Then 

\[ S(A, x^{\gamma - \beta}) = \lambda S(B, x^{\gamma - \beta}) + o\left( \frac{#A}{(\log x)^C} \right). \]

**Proof:** Repeated applications of Buchstab's identity.

\[ S(A, x^{\gamma - \beta}) = S(A, x) - \sum_{x^{\beta} \leq \rho < x^{\gamma - \beta}} S(A, \rho, \rho). \]

Good by Type I estimates.

\[ \text{Keep on applying Buchstab.} \]

To keep track of everything, it is convenient to write 

\[ S(C, x) = S(A_d, x) - \lambda S(B_d, x). \]

\[ T_n = \sum_{x^{\beta} \leq \rho \leq \ldots \leq \rho \leq x^{\gamma - \beta}} S(C, \rho, \rho). \]

\[ U_n = \sum_{x^{\beta} \leq \rho \leq \ldots \leq \rho \leq x^{\gamma - \beta}} S(C, \rho, x). \]

\[ V_n = \sum_{x^{\beta} \leq \rho \leq \ldots \leq \rho \leq x^{\gamma - \beta}} S(C, \rho, \rho). \]

Buchstab's identity: 

\[ T_n = U_n - T_{n+1} - V_{n+1}. \]

\[ \sum_{n=0}^{\infty} U_j (D^n) + \sum_{j=0}^{\infty} V_j (-D)^j. \]

\[ S(C, x^{\gamma - \beta}) = S(C, x) - \sum_{x^{\beta} \leq \rho < x^{\gamma - \beta}} S(C, \rho). \]
Proposition ( Vaughan's Identity )

Imagine $A$ satisfies a Type I estimate in $[x, \beta]$. 

Type II estimate if $\beta, \gamma$:

$$(x-\beta)^{-\alpha} \geq 1 \quad \text{then}$$

$$\# \{ p \in A \} = \# \{ p \in B \} + o\left(\frac{\#A}{\log x}\right).$$

Proof: Note: That $S(A_{\rho_{n}}, \rho_{n}, \rho_{n})$ counts products $\rho_{n} \rho_{i} \rho_{i} \in A_{\rho_{n}}$

with $\rho_{n} \geq \frac{x}{\rho_{n}}$, and $\rho_{n} \geq \frac{x}{\rho_{n}}$.

If $\rho_{n} \geq \frac{x}{\rho_{n}}$, then this forces $\rho_{n}$ to be prime.

Then $S(A_{\rho_{n}}, \rho_{n}, \rho_{n}) = S(A_{\rho_{n}}, (\frac{x}{\rho_{n}})^{\alpha})$.

In probab., $S(A_{\rho}, \rho) = S(A_{\rho}, \min(\rho, (\frac{x}{\rho})^{\alpha}))$ (for $\rho_{n} \approx \frac{x}{\rho}$).

$$\# \{ p \in A \} = S(A, (2x)^{\alpha}) $$

we want to show $S(A, (2x)^{\alpha}) = o\left(\frac{\#A}{\log x}\right)$.

$$S(C, (2x)^{\alpha}) = S(C \setminus (x^{2}p^{\alpha})) - \sum_{x^{2}p^{\alpha} \leq (2x)^{\alpha}} S(C, p).$$

Let $T_{n} = \sum_{x^{2}p^{\alpha} \leq (2x)^{\alpha}} S(C_{\rho_{n}}, \rho_{n}, \rho_{n})$

$$x^{2}p^{\alpha} \leq p_{n} \leq \ldots \leq p_{n} \leq (2x)^{\alpha}$$

$A_{\rho_{n}} \rho_{n} \leq \times U_{i}$

$$U_{n} = \sum_{x^{2}p^{\alpha} \leq p_{n} \leq (2x)^{\alpha}} S(C_{\rho_{n}}, x^{2}p^{\alpha})$$

$A_{\rho_{n}} \rho_{n} \leq \times U_{j}$

$$T_{i} = \sum_{j=1}^{\infty} (-1)^{j-i} U_{j}$$

since $S(C_{\rho_{n}}, \rho_{n}, \rho_{n}) = S(C_{\rho_{n}}, \min(\rho_{n}, (\frac{x}{\rho_{n}})^{\alpha}))$.

Buchtelb $\equiv T_{n} = U_{n} - T_{n+1}$

since $S(C_{\rho_{n}}, \rho_{n}, \rho_{n}) = S(C_{\rho_{n}}, \min(\rho_{n}, (\frac{x}{\rho_{n}})^{\alpha}))$.

$\therefore$ sufficient to show that $\# U_{n} = o(\frac{\#A}{\log x})$ for all $j$. 

...
But this follows from the same argument as previous proposition!

In \( U_n \), \( p_1 \cdots p_n \leq x \) and \( p_n \geq x^{1-\gamma} \)

\[ \Rightarrow p_1 \cdots p_n \leq x^{1-(\gamma-\beta)} \leq x^\alpha \quad \text{since} \quad \alpha+(\gamma-\beta) > 1. \]

Slight generalisation of previous proposition shows that

\[
\sum S(C_{p_1 \cdots p_n}, x^{1-\gamma}) = o\left(\frac{\#A}{\log x}\right).
\]

(prove linearity properties)

**Remarks.**

- If \( \alpha = 1-\varepsilon \) then you only need a Type I estimate \( \varepsilon \) width \( \varepsilon \) to break parity and get asymptotic formula for primes.

- This inclusion-exclusion approach naturally generalises to give non-trivial lower bounds even if \( \alpha+(\gamma-\beta) < 1 \) provided \( \varepsilon \) is big enough.

**Example:**

\[ \text{Type I: } \left[ 0, \frac{1}{3} \right] \quad \text{so} \quad \alpha+(\gamma-\beta) = 1-\varepsilon \]

\[ \text{Type II: } \left[ \frac{1}{3}, \frac{2}{3} \right] \quad \text{so} \quad \gamma-\beta = \varepsilon \]

\[ S(C_{x^{1/2}}) = S(C_{x^{1/3}}) - \sum_{x^{1/3} < p \leq x^{1/2}} S(C_{p}, x^{1/3}) - \sum_{x^{1/3} < p \leq x} S(C_{p}, x^{1/2}) \]

\[ = \sum_{x^{1/3} < p \leq x} S(C_{p}, x^{1/3}) = \sum_{x^{1/3} < p \leq x \log x} S(C_{p}, x^{1/3}) = \prod \text{product of primes all close to } x^{1/3} \quad \left( c(x^{1/3-2\varepsilon}, x^{1/3+2\varepsilon}) \right) \]

Since products of up to primes are in short intervals on a log-scale, we see

\[ S(C, x^{1/2}) \ll \frac{\#A}{\log x} \], so if \( S \) is small enough

\[ S(A, x^{1/2}) \gg \frac{\#A}{\log x}. \]

But **FACT:** \( S \) selects \( A \) which satisfies this Type I/II estimates but
have different asymptotics for the number of primes.