Type I/II sums and sieves

Motivating Q: Given $\mathcal{A} \subseteq [x, 2x]$, how many primes are in $\mathcal{A}$?

Examples: 1) $\mathcal{A} = \{ p+2 \in [x, 2x] : p \text{ prime} \}$ $\longleftrightarrow$ Twin primes

2) $\mathcal{A} = \{ n^2+1 : n \in [x^{1/2}, (2x)^{1/2}] \}$ $\longleftrightarrow$ Primes of the form n^2+1.

3) $\mathcal{A} = \{ n \in [x, 2x] : \|\alpha n\| \leq x^{-0.1} \}$ $\longleftrightarrow$ Prime in Bohr sets

(here α is a fixed irrational, $\|\cdot\| =$ distance to the nearest integer)

Expect asymptotic formulae in all cases, but only known for 3.

In general this question is v. difficult! But Type I/II sums + sieve methods provide a general framework for approaching these problems.

Arithmetic information about $\mathcal{A}$ $\longrightarrow$ $\boxed{\text{Sieve Methods}}$ $\longrightarrow$ Information about primes in $\mathcal{A}$

Sieve methods are <u>flexible</u> <u>combinatorial</u> techniques to turn arithmetic information that we can prove into arithmetic information we care about.

Q: Given $\mathcal{A}$ of interest, what arithmetic information can we prove about $\mathcal{A}$?

Q: How would we turn this into information about primes in $\mathcal{A}$?

Basic / traditional sieve methods just use 'Type I' arithmetic information.

$\Updownarrow$

Distribution of $\mathcal{A}$ in residue classes

Notation: $\mathcal{A}_d := \{ a \in \mathcal{A} : d \mid a \}$.

Examples ① $\mathcal{A} = \{ p+2 \in [x, 2x] \}$

Bombieri-Vinogradov $\Rightarrow$ $$\sum_{d \leq x^{\frac{1}{2}-\varepsilon}} \left| \#A_d - \frac{g(d)}{d}\#A \right| \ll_{A,\varepsilon} \frac{\#A}{(\log x)^A}.$$

where $g(d) = \begin{cases} \frac{d}{\phi(d)}, & \text{if } d \text{ odd.} \\ 0, & \text{if } d \text{ even.} \end{cases}$

Conjectured that we can get same result for $d \leq x^{1-\varepsilon}$.

② $A = \{n^2+1 \in [x, 2x]\}$

$$\sum_{d \leq x^{\frac{1}{2}-\varepsilon}} \left| \#A_d - \frac{g(d)}{d}\#A \right| \ll_{A,\varepsilon} \frac{\#A}{(\log x)^A}.$$

where $g(d) = \#\{m \in \mathbb{Z}/d\mathbb{Z} : m^2 \equiv -1 \ (d)\}$

This is essentially best-possible!

Looking at $n \in [x^{\frac{1}{2}}, (2x)^{\frac{1}{2}}]$ s.t. $n^2 \equiv -1 \ (d)$.

so there are $\frac{(2x)^{\frac{1}{2}} - x^{\frac{1}{2}}}{d} + O(1)$ elements $n \equiv \omega \pmod d$ for each $\omega \pmod d$ with $\omega^2 \equiv -1 \pmod d$.

③ $A = \{n \in [x, 2x] : \|\alpha n\| \leq x^{-0.1}\}$

Can get reasonable understanding of A in arithmetic progressions here too.

In these setups this arithmetic info gives satisfying answer to studying the number of elements of A with no small prime factors.

$$S(A, z) = \#\{a \in A : a \text{ has no prime factors} \leq z\}.$$

e.g. $S(A, (2x)^{\frac{1}{2}}) = \#\{\text{primes in } A\}$.

$S(A, x^{\frac{1}{100}}) = \#\{\text{elements of } A \text{ with no factor} \leq x^{\frac{1}{100}}\}$

$\leq \#\{\text{elements of } A \text{ with at most } 101 \text{ prime factors}\}$.

$$S(A, 6) = \#A - \#A_2 - \#A_3 - \#A_5 + \#A_6 + \#A_{10} + \#A_{15} - \#A_{30}$$

$\therefore S(A, z)$ is completely determined by $\#A_d$ for $d \mid \prod_{p \leq z} p$

~~But~~ This involves d which are way too big when $z \geq \log x$.

THM (Linear sieve). Let $A \subseteq [x, 2x]$ such that

$$\sum_{d \leq x^{\theta}} \left| \# A_d - \frac{g(d)}{d} \# A \right| \leq \frac{\# A}{(\log x)^{100}}$$

for some multiplicative function $g(d)$ with (A) $\sum_{p \leq y} \frac{g(p)}{p} = \sum_{p \leq y} \frac{1}{p} + O(1)$ ← "$g(p)$ is 1 on average"

(B) $g(p) \leq (1-\varepsilon)p$ ← "not all elements are a multiple of p"

Then

$$\mathfrak{S}\left(f\left(\tfrac{\theta}{c}\right) + o(1)\right) \frac{\# A}{c \log x} \leq \overset{S(A, x^c)}{\overset{\parallel}{\# \{\text{elements of } A \text{ with no factors} \leq x^c\}}} \leq \left(F\left(\tfrac{\theta}{c}\right) + o(1)\right) \frac{\# A}{c \log x} \mathfrak{S}$$

for some continuous functions f, F with

$$\mathfrak{S} = \prod_p \left(1 - \frac{g(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-1}$$

$F(s) = \frac{2}{s} \quad 1 \leq s \leq 3$

$f(s) = 0 \quad 0 \leq s \leq 2$

$\frac{\partial}{\partial s}\left(s F(s)\right) = f(s-1)$

$\frac{\partial}{\partial s}\left(s f(s)\right) = F(s-1)$

(Fact: $F(s), f(s) \to e^{-\gamma}$ as $s \to \infty$)

Amazing Fact: This is essentially best possible!

$\exists$ sets A^+ and A^- s.t.

$$\sum_{d \leq x^{1-\varepsilon}} \left| A_d^+ - \frac{g^+(d)}{d} \# A^+ \right| \ll_A \frac{\# A^+}{(\log x)^A}$$

$$\sum_{d \leq x^{1-\varepsilon}} \left| A_d^- - \frac{g^-(d)}{d} \# A^- \right| \ll_A \frac{\# A^-}{(\log x)^A}$$

$$\left(f\left(\tfrac{1}{c}\right) + o(1)\right) \frac{\# A^-}{c \log x} = S(A^-, x^c) \leq S(A^+, x^c) = \left(F\left(\tfrac{1}{c}\right) + o(1)\right) \frac{\# A}{c \log x}$$

for all $c \in (0,1)$.