Probabilistic aspects of character sums
Lecture 4: Fixed length, varying character

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Recall: r is a large prime, and χ a (non-principal) Dirichlet character mod r.

Plan of the talk:

- Introduction to the problem
- First thoughts about a random model
- Previous work
- Statement of results
- Analysing the random model
- "Derandomising" the random model

The problem

Investigate the statistical behaviour of

$$\sum_{n \le x} \chi(n),$$

where x is *fixed* (in terms of r) and χ varies over non-principal characters mod r.

We would like to understand the size of the sum for "most" characters χ, as well as the proportion of χ for which it attains unusually large or small values.

First thoughts

- By periodicity of $\chi(n)$ mod r, we only need to investigate $1 \leq x \leq r$.
- Our naive expectation might be that on this range (except perhaps if x is very close to r), we should usually have $|\sum_{n\leq x} \chi(n)| \approx \sqrt{x}$ ("squareroot cancellation").
- When $x \leq \sqrt{r}$, these are *short* sums, so we might be able to ignore periodicity and simply model $\sum_{n\leq x} \chi(n)$ by the sum $\sum_{n\leq x} f(n)$ of a Steinhaus random multiplicative function.
- When $x > \sqrt{r}$, we can use the "Fourier flip" $|\sum_{n\leq x} \chi(n)| \approx \frac{x}{\sqrt{r}} |\sum_{k\leq r/x} \chi(k)|$.

- To investigate $\sum_{n \le x} \chi(n)$, we will try to understand the *moments*
$$\frac{1}{r-2} \sum_{\chi \neq \chi_0 \bmod r} |\sum_{n \le x} \chi(n)|^{2q}.$$
- When q is large, good upper bounds for the moments might (or might not...) give good upper bounds for the proportion of χ for which $\sum_{n \le x} \chi(n)$ is unusually large.
- When q is small (*in fact when* $q < 1$), the moments tell us more about the typical size of $\sum_{n \le x} \chi(n)$.

Previous work

As we calculated in Lecture 2, for $x < r$ we have

$$\begin{aligned}
\frac{1}{r-1} \sum_{\chi \bmod r} \Big|\sum_{n \le x} \chi(n)\Big|^2 &= \frac{1}{r-1} \sum_{\chi \bmod r} \sum_{n,m \le x} \chi(n)\overline{\chi(m)} \\
&= \sum_{n,m \le x} \mathbf{1}_{n \equiv m \bmod r} \\
&= \lfloor x \rfloor
\end{aligned}$$

So the second moment is *consistent* with the sums $\sum_{n \le x} \chi(n)$ usually having size $\approx \sqrt{x}$.

One can do a similar calculation of $\frac{1}{r-1} \sum_{\chi \bmod r} |\sum_{n \le x} \chi(n)|^{2q}$ whenever $q \in \mathbb{N}$ and $x^q < r$.
But this won't work when $q \notin \mathbb{N}$, in particular when $0 < q < 1$.

Montgomery and Vaughan (1979): for any $q > 0$ and any $x \leq r$, we have

$$\frac{1}{r-2} \sum_{\chi \neq \chi_0 \bmod r} \Big|\sum_{n \leq x} \chi(n)\Big|^{2q} \ll_q r^q.$$

- This improves on the Pólya–Vinogradov inequality on average ($(\sqrt{r})^{2q}$ rather than $(\sqrt{r}\log r)^{2q}$).
- When $x = r/2$, say, this shows that for a positive proportion of χ mod r we have $|\sum_{n \leq x} \chi(n)| \asymp \sqrt{r} \asymp \sqrt{x}$ (compare second and fourth moments).
- **BUT** the bound does not improve when x gets smaller.

For Steinhaus random multiplicative functions, we also have (as seen in Lecture 2)

$$\mathbb{E}|\sum_{n\leq x} f(n)|^2 = \mathbb{E}\sum_{n,m\leq x} f(n)\overline{f(m)} = \sum_{n,m\leq x} \mathbf{1}_{n=m} = \lfloor x \rfloor.$$

Using Hölder's inequality, for any $0 \leq q \leq 1$ we get

$$\mathbb{E}|\sum_{n\leq x} f(n)|^{2q} \leq \mathbb{E}(|\sum_{n\leq x} f(n)|^2)^q \leq x^q.$$

(And one can do the same for $\sum_{n\leq x}\chi(n)$.)
But Helson conjectured that something stronger should be true.
Conjecture: (Helson, 2010) $\mathbb{E}|\sum_{n\leq x} f(n)| = o(\sqrt{x})$ as $x \to \infty$.

Statement of results

Theorem 1 (H., 2020)

If $f(n)$ is a Steinhaus random multiplicative function, then uniformly for all large x and real $0 \le q \le 1$ we have

$$\mathbb{E}|\sum_{n\le x} f(n)|^{2q} \asymp \left(\frac{x}{1+(1-q)\sqrt{\log\log x}} \right)^q .$$

In particular, $\mathbb{E}|\sum_{n\le x} f(n)| \asymp \frac{\sqrt{x}}{(\log\log x)^{1/4}}$. *"Better than squareroot cancellation"*

Theorem 2 (H.)

Let r be a large prime. Then uniformly for any $1 \leq x \leq r$ and $0 \leq q \leq 1$, if we set $L := \min\{x, r/x\}$ we have

$$\frac{1}{r-2} \sum_{\chi \neq \chi_0 \text{ mod } r} \Big|\sum_{n \leq x} \chi(n)\Big|^{2q} \ll \left(\frac{x}{1 + (1-q)\sqrt{\log\log 10L}} \right)^{q}.$$

Because of the "Fourier flip", Theorem 2 (involving L) is the natural analogue of the upper bound part of Theorem 1.

In particular, we have

$$\frac{1}{r-2} \sum_{\chi \neq \chi_0 \bmod r} \Big|\sum_{n \leq x} \chi(n)\Big| \ll \frac{\sqrt{x}}{(\log\log 10L)^{1/4}}.$$

So provided $x \to \infty$ but $x = o(r)$, the typical size of $\sum_{n \leq x} \chi(n)$ is $o(\sqrt{x})$ ("better than squareroot cancellation").

How is Theorem 1 proved?

- Show that for $0 \le q \le 1$ we have

$$\mathbb{E}|\sum_{n\le x} f(n)|^{2q} \approx x^q \mathbb{E}\left(\frac{1}{\log x}\int_{-1/2}^{1/2} |F(1/2+ih)|^2 dh\right)^q,$$

where $F(s) := \prod_{p\le x}(1-\frac{f(p)}{p^s})^{-1}$ is the random Euler product corresponding to $f(n)$.

- Show that
$\mathbb{E}\left(\frac{1}{\log x}\int_{-1/2}^{1/2} |F(1/2+ih)|^2 dh\right)^q \asymp \left(\frac{1}{1+(1-q)\sqrt{\log\log x}}\right)^q.$

Step 2: is the most interesting from a probabilistic point of view— there is a connection with an object called *critical multiplicative chaos*. But we won't discuss this much here.

Note that using Hölder's inequality, for $0 \le q \le 1$ we trivially have

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\log x}\int_{-1/2}^{1/2}|F(1/2+ih)|^2\right)^q &\le \left(\frac{1}{\log x}\int_{-1/2}^{1/2}\mathbb{E}|F(1/2+ih)|^2\right)^q \\ &\ll 1.\end{aligned}$$

To improve on this, one should restrict to the case where $|F(1/2+ih)|$ *and its partial products* satisfy certain size bounds at a net of points h (and show that such a restriction is permissible).

Step 1: uses conditioning, i.e. "freezing" some of the randomness and only working with what is left.

For simplicity, we only discuss the upper bound and only look at part of the sum:

$$\sum_{\substack{n\le x, \\ P(n)>\sqrt{x}}} f(n),$$

where $P(n)$ denotes the largest prime factor of n.

Since $f(n)$ is multiplicative, we have

$$\sum_{\substack{n\le x, \\ P(n)>\sqrt{x}}} f(n) = \sum_{\sqrt{x}<p\le x} f(p) \sum_{m\le x/p} f(m).$$

Let $\mathbb{E}^{(\sqrt{x})}$ denote expectation conditional on the values $(f(p))_{p \leq \sqrt{x}}$. Then

$$\begin{aligned} \mathbb{E}\Bigg| \sum_{\substack{n \leq x, \\ P(n) > \sqrt{x}}} f(n) \Bigg|^{2q} &= \mathbb{E}\mathbb{E}^{(\sqrt{x})} \Bigg| \sum_{\substack{n \leq x, \\ P(n) > \sqrt{x}}} f(n) \Bigg|^{2q} \\ &\leq \mathbb{E}\left(\mathbb{E}^{(\sqrt{x})} \Bigg| \sum_{\substack{n \leq x, \\ P(n) > \sqrt{x}}} f(n) \Bigg|^{2} \right)^{q}, \end{aligned}$$

by (conditional) Hölder's inequality.

And the right hand side is $= \mathbb{E}\left(\sum_{\sqrt{x} < p \leq x} \left| \sum_{m \leq x/p} f(m) \right|^{2} \right)^{q}$.

Since primes $\sqrt{x} < p \leq x$ are quite well distributed, with density $\asymp \frac{1}{\log x}$, our expectation is

$$\begin{aligned} &\approx \mathbb{E}\left(\frac{1}{\log x}\int_{\sqrt{x}}^{x}\left|\sum_{m \leq x/t} f(m)\right|^2 dt\right)^q \\ &= x^q \mathbb{E}\left(\frac{1}{\log x}\int_{1}^{\sqrt{x}}\left|\sum_{m \leq z} f(m)\right|^2 \frac{dz}{z^2}\right)^q. \end{aligned}$$

By a multiplicative version of Parseval's identity, the right hand side is

$$\approx x^q \mathbb{E}\left(\frac{1}{\log x}\int_{-1/2}^{1/2} |F(1/2+ih)|^2 dh\right)^q,$$

as claimed.

Can Step 1 work for Dirichlet characters?

We need an analogue of conditioning for Dirichlet characters.

This seems possible: break up $\frac{1}{r-1}\sum_{\chi \bmod r}$ into subsums depending on the behaviour of χ.

For example, let $\epsilon > 0$ be a small parameter, and split up the characters into classes according to the values on the unit circle (discretised into arcs of length $2\pi\epsilon$) taken by $(\chi(p))_{p\leq\sqrt{x}}$. (Recall the proof of the lower bound in Lecture 2.)

BUT: there is a big problem with this when x is moderately large. Given any arcs I_p of length $2\pi\epsilon$, we have

$$\mathbb{P}(f(p) \in I_p \; \forall \; p \le \sqrt{x}) = \prod_{p \le \sqrt{x}} \mathbb{P}(f(p) \in I_p) \approx \epsilon^{2\sqrt{x}/\log x}.$$

If $x \gg (\log r \log\log r)^2$, this probability will be smaller than $1/r^C$.

So it is impossible for the proportion of characters χ mod r satisfying $\chi(p) \in I_p$ for all $p \le \sqrt{x}$ to closely approximate $\mathbb{P}(f(p) \in I_p \; \forall \; p \le \sqrt{x})$, as that proportion can only assume the values

$$0, \frac{1}{r-1}, \frac{2}{r-1}, \frac{3}{r-1}, \ldots$$

Derandomising the random proof

Recall that $L := \min\{x, r/x\}$.

There are two observations we can exploit to save the situation:

- log log is a very slowly growing function, so rather than "conditioning" on $\chi(p)$ for all $p \leq L$ we might only need to look at $p \leq P := \exp\{\log^{1/3} L\}$, say.

$$\log\log P = \log(\log^{1/3} L) \asymp \log\log L.$$

- we don't really need to fix all the values $(\chi(p))_{p \leq P}$. We only need to condition on enough data to (roughly) fix the quantities required in the proof of Theorem 2.

Idea: Condition (i.e. break up $\frac{1}{r-1}\sum_{\chi \text{ mod } r}$ into subsums) according to the values taken by

$$|\prod_{p \leq P}(1 - \frac{\chi(p)}{p^{1/2+ik/\log P}})^{-1}|, \quad |k| \leq \frac{\log P}{2}.$$

Roughly speaking, knowing these values is enough to approximately fix the size of

$$\frac{1}{\log P}\int_{-1/2}^{1/2}|\prod_{p \leq P}(1 - \frac{\chi(p)}{p^{1/2+ih}})^{-2}|dh.$$

And now we are only conditioning on $\asymp \log P$ values (rather than $\asymp P$ values), so each subsum will still contain a substantial numbers of characters to average over.

Key point: As χ varies mod r, the distribution of the values

$$|\prod_{p \le P}(1 - \frac{\chi(p)}{p^{1/2+ik/\log P}})^{-1}|, \quad |k| \le \frac{\log P}{2}$$

is very close to the distribution of the values

$$|F(1/2 + ik/\log P)|, \quad |k| \le \frac{\log P}{2}$$

from the random case (with the Euler product of length P).

One can prove this rigorously by comparing moments of

$$\Re \sum_{p \le P} \frac{\chi(p)}{p^{1/2+ik/\log P}} \quad \text{and} \quad \Re \sum_{p \le P} \frac{f(p)}{p^{1/2+ik/\log P}}.$$

This exploits the fact that $P = \exp\{\log^{1/3} L\}$ *is always much smaller than* r*, so periodicity doesn't come into play.*

At the end, one deduces (very roughly) that

$$
\begin{aligned}
& \frac{1}{r-2} \sum_{\chi \neq \chi_0 \bmod r} \Big|\sum_{n \leq x} \chi(n)\Big|^{2q} \\
\leq \quad & \frac{1}{r-2} \sum_{\chi \neq \chi_0 \bmod r} x^q \mathbb{E}\left(\frac{1}{\log P} \int_{-1/2}^{1/2} \Big| \prod_{p \leq P} \Big(1 - \frac{\chi(p)}{p^{1/2+ih}}\Big)^{-2} \Big| dh \right)^q \\
\approx \quad & \mathbb{E} x^q \left(\frac{1}{\log P} \int_{-1/2}^{1/2} |F(1/2+ih)|^2 dh \right)^q,
\end{aligned}
$$

with the random Euler product of length P.

So one can use the proof from the random case to finish the proof in the deterministic case!

Open problem (probably hard): obtain a matching lower bound in Theorem 2.

By a different combinatorial method, La Bretèche, Munsch and Tenenbaum recently proved that for $1 \leq x < r/2$,

$$\frac{1}{r-2} \sum_{\chi \neq \chi_0 \bmod r} \Big| \sum_{n \leq x} \chi(n) \Big| \gg \frac{\sqrt{x}}{\log^{c+o(1)} x}, \qquad c \approx 0.04304.$$

If one could obtain a lower bound that matched Theorem 2 (for $x \leq r^{1/2+o(1)}$), this would (essentially) imply a positive proportion non-vanishing result for Dirichlet theta functions $\theta(1;\chi)$.