Riemann ζ

- In this lectures I will discuss the basic theory of the Riemann ζ function.
- The plan for the four lectures is as follows:
  1. Main properties
  2. Computing and bounding the Riemann zeta-function
  3. Zero-density theorems and mollifiers
  4. The finer aspects

The Riemann ζ-function

- The Riemann ζ function is defined in $\Re s > 1$ as

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}.$$

- This defines an analytic function in the region $\Re s > 1$.
- Because every integer n can be factored uniquely into prime factors we can also write,

$$\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

where the product is taken over all primes p.

Infinitude of primes

- The two representations

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)$$

can be used to obtain the divergence of the series

$$\sum_p \frac{1}{p}.$$

- Indeed take $s > 1$ to be real. Then,

$$\sum_{n \geq 1} \frac{1}{n^s} = \int_1^{\infty} x^{-s} dx + O(1) = \frac{1}{s-1} + O(1)$$

- On the other hand

$$\log \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1} = \sum_p \frac{1}{p^s} + O(1)$$

Infinitude of primes

- Putting these two together we conclude

$$\sum_p \frac{1}{p^s} = \log\Big(\frac{1}{s-1} + O(1)\Big) + O(1)$$

- Which gives the divergence of $\sum_p \frac{1}{p}$.
- Taking $s = 1 + \frac{1}{\log x}$ suggests that

$$\sum_{p \le x} \frac{1}{p} \approx \sum_p \frac{1}{p^s} = \log\log x + O(1)$$

- With more work this can be established.
- Incidentally we can also get the infinitude of the primes from

$$\prod_p \Big(1 - \frac{1}{p^2}\Big)^{-1} = \sum_{n \ge 1} \frac{1}{n^s} = \frac{\pi^2}{6} \notin \mathbb{Q}$$

Analytic continuation

- It is clear that there is content in playing these two representations against each other.
- A natural step further is to try to analytically continue the Riemann zeta-function to the whole complex plane.
- Notice that,

$$\Gamma(s)\frac{1}{n^s} = \int_0^\infty e^{-nx}x^{s-1}dx$$

- Summing this over $n \geq 1$ in $\Re s > 1$ we get

$$\Gamma(s)\zeta(s) = \int_0^\infty \Big(\sum_{n\geq 1} e^{-nx}\Big)x^{s-1}dx = \int_0^\infty \frac{e^{-x}}{1-e^{-x}} \cdot x^{s-1}dx$$

Analytic continuation

- It therefore remains to analytically continue

$$\left(\int_0^1 + \int_1^\infty\right) \frac{e^{-x}}{1 - e^{-x}} x^{s-1} dx =$$

  to the whole complex plane.

- In the integral over $x \in [1, \infty)$ we simply integrate by parts repeatedly. This gives a meromorphic continuation to $\Re s > -A$ for any given $A > 0$.

- In the integral over $x \in [0, 1)$ we expand $\frac{e^{-x}}{1-e^{-x}}$ into a Taylor series and obtain the requisite *meromorphic* continuation to $\Re s > -A$ for any given $A > 0$.

- The only possible poles are at $s \in \{1, 0, -1, -2, \ldots\}$ and by being careful we can show that only the pole at $s = 1$ actually exists.

The explicit formula

- Since $\zeta(s)$ admits an meromorphic continuation to the whole complex plane $\mathbb{C}$ so does

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha \\ 0 & \text{otherwise} \end{cases}$$

- Can we exploit this to say something deeper about the distribution of prime numbers?

Mellin transforms

- We make a small aside on *Mellin transforms*
- Let W a Schwartz function, compactly supported in $(0, \infty)$.
- The Mellin transform,

$$\widetilde{W}(x) = \int_0^\infty W(x) x^{s-1} dx$$

  is an entire function such that $W(\sigma + it) \ll_{\sigma,A} (1 + |t|)^{-A}$ for any given A and any σ in a fixed strip.
- We also have the inverse Mellin transform

$$W(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} \widetilde{W}(s) x^{-s} dx$$

- These are multiplicative analogues of the usual Fourier transform (or in this context more precisely Laplace transform)

The explicit formula

- By Mellin inversion we have,

$$W\Big(\frac{n}{x}\Big) = \frac{1}{2\pi}\int_{2-i\infty}^{2+i\infty} \widetilde{W}(s)\Big(\frac{x}{n}\Big)^{s} dx$$

- Summing over n with weights $\Lambda(n)$ we get

$$\sum_{n \geq 1} \Lambda(n) W\Big(\frac{n}{x}\Big) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Big(- \frac{\zeta'}{\zeta}(s)\Big) \widetilde{W}(s) x^{s} ds$$

- Let's take for granted that $\frac{\zeta'}{\zeta}(s) \ll (1 + |s|)^{A}$ for some fixed constant $A > 0$. We will show this later.
- If that is the case then we can shift the contour from the line $2 + i\mathbb{R}$ to the line $-\varepsilon + i\mathbb{R}$. When doing so we collect residues from the poles of ζ'/ζ.

The explicit formula

- Doing so gives us the explicit formula

$$\sum_{n\geq 1} \Lambda(n) W\left(\frac{n}{x}\right) = x + \sum_{\rho} x^{\rho} \widetilde{W}(\rho)$$

where ρ goes over the zeros of the Riemann zeta function and The term x comes from the simple pole at $s = 1$.

- This makes it clear that $\sum_{n\leq x} \Lambda(n) \sim x$ is equivalent to $\zeta(s)$ not having any zeros on $\Re s = 1$.
- Let's W tend to the indicator function of $[0, 1]$, then we get

$$\sum_{n\leq x} \Lambda(n) = x + \sum_{\rho} \frac{x^{\rho}}{\rho}$$

- This immediately shows that there are infinitely many zeros of $\zeta(s)$. The left-hand side is discontinuous, but if there were only finitely many zeros the right hand side would be continuous.

The explicit formula

- Furthermore if we believe that primes are "like a random sequence" then we would expect that

$$\sum_{n \le x} \Lambda(n) \approx x + O(\sqrt{x})$$

- This together with the explicit formula suggests that all the zeros of $\zeta(s)$ are located in $\Re s \le \frac{1}{2}$.
- However one cannot extract too much non-trivial information from the explicit formula. The explicit formula is the statement that zeros and primes are equivalent. But not much else.

Limitations of the explicit formula

- One good way of understanding the limitations of the explicit formula is to take the difference between the expression for x and $x+h$, getting

$$\sum_{x \leq n \leq x+h} \Lambda(n) = h + \sum_{\rho} \frac{(x+h)^\rho - x^\rho}{\rho}$$

- Roughly speaking

$$\frac{(x+h)^\rho - x^\rho}{\rho} \approx \begin{cases} hx^{\rho-1} & \text{if } |\rho| \leq x/h \\ 0 & \text{otherwise} \end{cases}$$

- So that

$$\sum_{x<n<x+h} \Lambda(n) \approx h + h \sum_{|\rho| \leq x/h} x^{\rho-1}$$

- So if we want to check whether n is prime that would require us to compute all the zeros up to height n.

Limitations

- Conversely if we wanted to know very precise information about zeros it would require a lot of information about primes.
- This is simply a form of Heisenberg's uncertainty principle for the Fourier transform.
- The explicit formula cannot give us simultaneously very precise information about the primes and the zeros. It's always either one or the other.
- Still the explicit formula is not useless:
  1. There are algorithms that determine all the zeros of $\zeta(s)$ up to height T in time T (relying on the FFT).
  2. By carefully balancing the explicit formula we get an algorithm for computing the number of primes up to x in time $\sqrt{x}$.

Further properties inside the complex plane

- We've seen so far that $\zeta(s)$ continue meromorphically to $\mathbb{C}$.
- Assuming that we can obtain bounds for $\frac{\zeta'}{\zeta}(s)$ away from zeros this meromorphic continuation is useful and relates the behavior of the primes to the location of the zeros of $\zeta(s)$.
- We need therefore to better understand the behavior of $\zeta(s)$ inside the complex plane.

Properties inside the complex plane

- Note that so far everything that we used came from the meromorphic continuation and the Euler product.
- Besides the Euler product the second deep property that the Riemann zeta-function possesses is the functional equation: if we let

$$\xi(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)$$

  then

$$\xi(s) = \xi(1-s).$$

What is the meaning of the functional equation

- Before sketching the proof of the functional equation, let me explain its meaning and consequences.
- The Euler product captures the fact that integers factor uniquely into prime numbers, i.e the multiplicative property of the integers
- The functional equation in turns captures the fact that integers form lattice, i.e the additive property of the integers. I will now explain why this is so.

Poisson summation

- We will show that the functional equation for the Riemann ζ function is equivalent to the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)$$

  which is valid for any Schwartz function f.

- A Poisson summation formula can hold only when there is an underlying (often quite hidden) lattice structure.
- In that sense the functional equation captures the fact that integers form a lattice.

Poisson summation

- Let me now quickly explain how this equivalence goes.
- In one direction if the Poisson summation formula holds then we specialize to $f(n) = e^{-n^2/x}$ for example, and we integrate both sides with respect to $\int_0^\infty (\ldots) x^{s-1} dx$. This gives the functional equation for $\zeta(s)$.
- We pick $f(n) = e^{-n^2/x}$ because it has nice transformation properties (in fact $\sum_{n \in \mathbb{Z}} e^{-n^2 z}$ is an automorphic form). But pretty much any other choice would have worked too; it would have simply led to more convoluted calculations.

Poisson summation

- In the other direction assume f is such that $f(0) = 0$ and f is even.
- We consider,

$$\sum_{n \geq 1} f(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \widetilde{f}(s)\zeta(s)ds$$

where

$$\widetilde{f}(s) = \int_0^\infty f(x)x^{s-1}dx.$$

- We now shift contours to the line $\Re s = -\varepsilon$. Thus

$$\sum_{n \geq 1} f(n) = \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \widetilde{f}(s)\zeta(s)ds.$$

- We now apply the functional equation in the form,

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} \zeta(1-s)$$

Poisson summation

- We find that

$$\sum_{n\geq 1} f(n) = \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \widetilde{f}(s)\pi^{s-1/2}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\zeta(1-s)ds$$

- We make the change of variable $s \mapsto 1-s$. We then get,

$$\frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \widetilde{f}(1-s)\pi^{1/2-s}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(s)ds$$

- We expand $\zeta(s)$ point-wise finding that the above is equal to

$$\sum_{n\geq 1} f^{\star}(n)$$

where

$$f^{\star}(x) := \frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \widetilde{f}(1-s)\pi^{1/2-s}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}x^{-s}ds$$

Poisson summation

- Unsurprisingly one can show (using a version of Plancherel for Mellin transforms) that

$$f^\star(x) = \int_{\mathbb{R}} f(y)\cos(2\pi xy)dy$$

- And thus we have obtained

$$\sum_{n\geq 1} f(n) = \sum_{n\geq 1}\int_{\mathbb{R}} f(y)\cos(2\pi ny)dy$$

which is an equivalent form of Poisson summation formula for any function f such that $f(0) = 0$ and f is even.

Consequences of the functional equation

- The functional equation has many important immediate consequences.
- First, there are no zeros in $\Re s < 0$ except at $s = -k$ with $k \in \mathbb{Z}$.
- Second,
$$\frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(1-s) + O(\log t)$$
In particular we have $\frac{\zeta'}{\zeta}(s) \ll \log t$ in $\Re s < 0$, justifying our previous assumption when deriving the explicit formula.
- Finally, we can trivially bound $\zeta(s)$ for $\Re s > 1$ and thus the functional equation gives us bounds for $\zeta(s)$ in $\Re s < 0$. Using convexity in complex analysis this gives us bounds for $\zeta(s)$ in the so-called *critical strip* $0 < \Re s < 1$.

Further consequences of the functional equation

- Another consequence of the functional equation is that

$$\zeta(\tfrac{1}{2} + it) = e^{-2i\theta(t)}\zeta(\tfrac{1}{2} - it)$$

where

$$e^{2i\theta(t)} := \pi^{it}\frac{\Gamma(\frac{1}{4} - \frac{it}{2})}{\Gamma(\frac{1}{4} + \frac{it}{2})}$$

- In particular this means that

$$Z(t) = e^{-i\theta(t)}\zeta(\tfrac{1}{2} - it) \in \mathbb{R}.$$

- No such normalization is known on any other line $\sigma + it$ with $\sigma \neq \frac{1}{2}$. This alones makes it much more likely for zeros of $\zeta(s)$ to occur on $\Re s = \frac{1}{2}$ since the function can be made real on this line and thus a zero comes simply from a sign change.

Further consequences of the functional equation

- In fact it is conjectured that the functional equation alone should be responsible for the claim that "100%" of the zeros of the Riemann ζ function lie on the critical line.

Summary

- There are only two important properties of the Riemann zeta-function : the Euler product and the functional equation.
- Everything that we know about the Riemann zeta function comes from one or the other.
- Usually the Euler product is used when talking about the zeros of the Riemann zeta-function.
- Usually the functional equation is used when we are interested in bounding the size of the Riemann zeta-function in the strip.
- In Lecture 2 I will discuss the consequences of the functional equation
- In Lecture 3 I will discuss the consequences of the Euler product
- In Lecture 4 we will study the finer aspects of the behavior of the Riemann zeta-function