Type I/II sums and sieves

Recap: \[ \#A_d = \frac{g(d)}{d} \#A \quad \text{"Type I estimate" for } d < \epsilon \]

Then \[ \frac{1}{C} \left( \frac{8(\epsilon^2)}{\log x} \right)^{\frac{A}{C}} S(\epsilon, x^c) \leq \left( \frac{F(x)}{\log x} \right)^{\alpha(1)} \frac{\#A}{\log x} \cdot C \] with \( C = \prod \left( 1 - \frac{\rho}{1 - \rho} \right) \).

for continuous \( S(\epsilon, f) \Rightarrow e^\gamma \).

Consequences: Cannot hope to show there exists primes in \( A \) using only Type I information

(union bound is optimal, but even if \( \Theta = 1 \) lower bound is trivial when \( \epsilon = \frac{1}{2} \)).

- Our Type I information for examples last time means

\[ \#\text{primes } x^\frac{1}{3} \ll \frac{x}{(\log x)^2} \]

\[ \#\text{primes } x^\frac{1}{6} \ll \frac{x}{(\log x)^2} \]

- Asymptotic formulae for \( S(\epsilon, x^c) \) as \( \epsilon \to 0 \)

\[ \#p \in (x, 2x): p \nmid 2 \text{ has no prime factors } \ll 3 = (1 + o(1)) \frac{e^\gamma x}{(\log x)^2} \cdot \Theta \]

where \( \Theta = \prod \left( 1 - \frac{\rho}{1 - \rho} \right) \left( 1 - \frac{1}{p} \right) \)

\( \Rightarrow \) only many primes \( p \) s.t. \( p \nmid 2 \) has \( 0(1) \) prime factors.

**PRINCIPLE:**

* Understanding \( A \) in \( AB \) is not enough to show there are primes in \( A \)

* If you understand \( A \) in \( AB \) then

\( \Rightarrow \) Upper bound (which should be \( \Theta \) correct order of magnitude) for \( \#p \in (x, 2x) \)

\( \Rightarrow \) Positive lower bound (of correct order of magnitude) for numbers with no prime factors \( \leq \epsilon x \) (so with at most \( \frac{1}{3} \) prime factors)

**OPEN PROBLEM:** Understand the situation when \( g(p) = \kappa \) for some \( \kappa > 1 \).

(In this case we have upper/lower bound functions \( \delta_x, \delta_x \), but no idea of optimality).

\[ \#p \leq x^{-\frac{1}{2}} \quad \text{for } n \in \{ n, n+1, \ldots, n+\kappa \} \text{ all prime } \leq (1 + o(1)) \frac{2^k x}{(\log x)^k} \cdot \Theta \]

where \( \Theta = \prod \left( 1 - \frac{\kappa}{1 - \kappa} \right) \left( 1 - \frac{1}{p} \right) \).
Is it necessary in this band?

(Very sketchy) Proof: Buchstabs identity. If \( z \geq z_2 \)

\[
S(A, z_1) = S(A, z_2) - \sum_{z_2 < p \leq z} S(A, p) \\
\uparrow \quad \uparrow \quad \uparrow \\
\text{elements with smallest prime factor } > z_2 \quad \text{elements with smallest prime factor } \leq z_2 \quad \text{prime factor in } (z, 2z).
\]

Proof of linear sieve bounds reeks of Buchstabs identity and \( S(A, z) \geq 0 \).

\[
S(A, z) = \#A - \sum_{p \leq z} S(A, p) \\
\leq \#A - \sum_{p \leq z} S(A, p) \quad \text{for } z \leq z_2 \\
= \#A - \sum_{p \leq z} \#A_p + \sum_{p < p, p \leq z} S(A, p, p).
\]

Key insight: Use the bound \( S(A, p_{\alpha_{j1}}, p_{\alpha_{j1}}) \geq 0 \) for an upper bound if \( p_{\alpha_{j1}} > x^\theta \).

Use Buchstabs identity

\[
S(A, p_{\alpha_{j1}}, p_{\alpha_{j1}}) = \#A_{p_{\alpha_{j1}}} - \sum_{p_{\alpha_{j1}} < p, p_{\alpha_{j1}}} S(A, p). \\
\]

Use Buchstabs identity if even number \( p_{\alpha_{j1}} < x^\theta \).

Q: How can we improve on this? What extra arithmetic information would allow us to count primes?

\[
\#_{\text{pe} \in A^3} = S(A, (2x)^{\frac{1}{2}}) = S(A, x^\epsilon) - \sum_{x^\epsilon \leq p \leq (2x)^{\frac{1}{2}}} S(A, p). \\
\uparrow \quad \uparrow \\
\text{Good understanding through Type F estimates.} \quad \text{Counts \textit{ae}A which can be written as } a = mm_2 \text{ with } m_1, m_2 \geq x^\epsilon.
\]
We can rewrite the bilinear sum as

$$\alpha^T \mathbf{M}^{(A)} \mathbf{B}$$

where

$$\mathbf{M}^{(A)}_{ij} = \begin{cases} 1, & \text{if } i, j \in A \\ 0, & \text{otherwise} \end{cases}$$

$$\alpha = \begin{pmatrix} \alpha_{m, p} \\ \vdots \\ \alpha_{m, p} \end{pmatrix}_{m \in \mathbb{P}, 2 \mathbb{P}}$$

$$\mathbf{B} = \begin{pmatrix} \beta_{m, p} \\ \vdots \\ \beta_{m, p} \end{pmatrix}_{m \in \mathbb{P}, 2 \mathbb{P}}$$

Amazing Fact: We have techniques that can estimate bilinear sums well without knowing anything about the coefficients $\alpha, \mathbf{B}$.

If we can understand these bilinear sums (by understanding $\mathbf{M}^{(A)}$), then we can understand $\sum S(A_p, p)$ and so $\# \mathcal{E} \mathcal{P} \mathcal{A}^3$. 

$$\sum_{x^2 \leq p \leq (2x)^\frac{3}{2}} \sum_{q \leq x^2 \leq \sqrt{p}} \sum_{x \leq p \leq (2x)^\frac{3}{2}} \sum_{r \leq x \leq \sqrt{p}} \sum_{p(r) \geq r} \sum_{p(q) \geq q} \sum_{p(e) \geq e} \sum_{p(f) \geq f} \sum_{p(g) \geq g}$$