Type I / II sums and sieves

Recap: If $\#\mathcal{A}_d \approx \frac{g(d)}{d}\#\mathcal{A}$ 'Type I estimate' for $d<x^\theta$ $\left(\begin{array}{c} g \text{ multiplicative} \\ g(p)\approx 1\end{array}\right)$

then $\mathfrak{S}\left(f\left(\frac{\theta}{c}\right)+o(1)\right)\frac{\#\mathcal{A}}{c\log x} \leq S(\mathcal{A},x^c) \leq \left(F\left(\frac{\theta}{c}\right)+o(1)\right)\frac{\#\mathcal{A}}{c\log x}\cdot\mathfrak{S}$ with $\mathfrak{S}=\prod_p\left(\frac{1-\frac{g(p)}{p}}{1-\frac{1}{p}}\right)$.

for continuous $f(s), F(s)\to e^{-\gamma}$.

Consequences: - Cannot hope to show there exists primes in $\mathcal{A}$ using only Type I information (lower bound is optimal, but even if $\theta\approx 1$ lower bound is trivial when $c=\frac{1}{2}$).

- Our Type I information for examples last time means:

$$\#\{\text{twin primes} \leq x\} \ll \frac{x}{(\log x)^2}$$

$$\#\{\text{primes of the form } n^2+1 \text{ in } [x,2x]\} \ll \frac{x^{1/2}}{(\log x)}.$$

- Asymptotic formulae for $S(\mathcal{A},x^\varepsilon)$ as $\varepsilon\to 0$.

$$\#\{p\in[x,2x]:\ p+2 \text{ has no prime factors} \leq x^\varepsilon\} = (1+O(\varepsilon))\frac{e^{-\gamma}x}{\varepsilon(\log x)^2}\mathfrak{S}$$

where $\mathfrak{S}=\prod_p\left(1-\frac{g(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1}$

$\Rightarrow$ ∞ly many primes p s.t. $p+2$ has $O(1)$ prime factors.

PRINCIPLE: * Understanding $\mathcal{A}$ in APs is not enough to show there are primes in $\mathcal{A}$

* If you understand $\mathcal{A}$ in APs then

$\Rightarrow$ Upper bound (which should be of correct order of magnitude) for $\#\{p\in\mathcal{A}\}$

$\Rightarrow$ Positive lower bound (of correct order of magnitude) for numbers with no prime factors $\leq x^\varepsilon$ (so with at most $1/\varepsilon$ prime factors)

OPEN PROBLEM: Understand the situation when $g(p)\approx\kappa$ for some $\kappa>1$.

(In this case we have upper/lower bound functions f_κ, F_κ, but no idea of optimality).

E.g. $\#\{n\leq x:\ n+h_1, n+h_2, \dots, n+h_k \text{ all prime}\} \leq (1+o(1))\frac{2^k k!\, x}{(\log x)^k}\mathfrak{S}$ $\quad h_1,\dots,h_k$ fixed.

where $\mathfrak{S}=\prod_p\left(1-\frac{\#\{h_1,\dots,h_k \pmod p\}}{p}\right)\Big/\left(1-\frac{k}{p}\right)$.

Is $k!$ necessary in this bound?

(Very sketchy) proof: Buchstab's identity: if $z_1 > z_2$

$$S(A, z_1) = S(A, z_2) - \sum_{z_2 < p \le z_1} S(A_p, p)$$

elements with smallest prime factor $> z_1$

elements with smallest prime factor $> z_2$

elements with smallest prime factor in $(z_1, z_2]$.

Proof of linear sieve bounds rests on Buchstab's identity and $S(A,z) \geq 0$.

$$S(A,z) = \#A - \sum_{p \le z} S(A_p, p)$$

$$\le \#A - \sum_{p \le z_1} S(A_p, p) \qquad \text{for } z_1 \le z$$

$$= \#A - \sum_{p \le z_1} \#A_p + \sum_{p_2 \le p_1 \le z} S(A_{p_1 p_2}, p_2)$$

Key insight: Use the bound $S(A_{p_1 \cdots p_{2j+1}}, p_{2j+1}) \geq 0$ for an upper bound if $p_1 \cdots p_{2j} p_{2j+1}^3 \geq x^\theta$.

Use Bushstab's Identity

$$S(A_{p_1 \cdots p_{2j+1}}, p_{2j+1}) = \#A_{p_1 \cdots p_{2j+1}} - \sum_{p_{2j+2} \le p_{2j+1}} S(A_{p_1 \cdots p_{2j+2}}, p_{2j+2})$$

if $p_1 \cdots p_{2j} p_{2j+1}^3 \le x^\theta$.

Use Buchstab's Identity if even number $p_1 \cdots p_{2j}$ of primes.

Q: How can we improve on this? What extra arithmetic information would allow us to count primes?

$$\#\{p \in A\} = S(A, (2x)^{1/2}) = S(A, x^\varepsilon) - \sum_{x^\varepsilon \le p < (2x)^{1/2}} S(A_p, p).$$

Good understanding just through Type I estimates.

Counts $a \in A$ which can be written as $a = m_1 m_2$ with $m_1, m_2 \geq x^\varepsilon$.

$$\sum_{x^\varepsilon \le p \le (2x)^{1/2}} S(A_p, p) = \sum_{x^\varepsilon \le p \le (2x)^{1/2}} \sum_{\substack{q \in [\frac{x}{p}, \frac{2x}{p}] \\ pq \in A \\ P^-(q) \ge p}} 1 \approx \sum_{\substack{P = 2^j \\ x^\varepsilon \le P \le (2x)^{1/2}}} \sum_{p \in [P, 2P]} \sum_{\substack{q \in [\frac{x}{P}, \frac{2x}{P}] \\ P^-(q) \ge P \\ pq \in A}} 1$$

$P^-(q)$ ↑ smallest prime factor

$$\approx \sum_{P} \underbrace{\sum_{m_1 \in [P, 2P]} \alpha_{m_1, P} \sum_{\substack{m_2 \in [\frac{x}{P}, \frac{2x}{P}] \\ m_1 m_2 \in A}} \beta_{m_2, P}}$$ ← Bilinear sum

where $\alpha_{m_1, P} = \mathbb{1}_{m_1 \text{ prime}}$

$\beta_{m_2, P} = \mathbb{1}_{P^-(m_2) \ge P}$.

Amazing Fact: We have techniques that can estimate bilinear sums well without knowing anything about the coefficients $\underline{\alpha}$, $\underline{\beta}$.

We can rewrite the bilinear sum as $\underline{\alpha}^T M^{(A)} \underline{\beta}$

where $M^{(A)}_{ij} = \begin{cases} 1, & \text{if } ij \in A \\ 0, & \text{o/w} \end{cases}$

$$\underline{\alpha} = \begin{pmatrix} \alpha_{m_1, P} \end{pmatrix}_{m_1 \in [P, 2P]} \qquad \underline{\beta} = \begin{pmatrix} \beta_{m_2, P} \end{pmatrix}_{m_2 \in [\frac{x}{P}, \frac{2x}{P}]}.$$

* If we can understand these bilinear sums (by understanding $M^{(A)}$) then we can understand $\sum S(A_p, p)$ and so $\#\{p \in A\}$ *.