Probabilistic aspects of character sums
Lecture 2: The maximum

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Recall: r is a large prime, χ is a (non-principal) Dirichlet character mod r.

Plan of the talk:

- Introduction to the maximum problem
- First thoughts (in light of PFE)
- Random multiplicative functions
- Distribution of the maximum
- Further thoughts

The problem

Investigate the statistical behaviour of

$$M(\chi) := \max_{1 \le x \le r} \Big| \sum_{n \le x} \chi(n) \Big|$$

as χ varies over non-principal characters mod r.

We would like to understand the proportion of χ for which $M(\chi)$ attains certain sizes (in terms of r).

(We might also try to investigate the structure of large values, e.g. is $M(\chi)$ largest for certain special χ?)

First thoughts

- Recall that we have the Pólya–Vinogradov bound

$$\begin{aligned} M(\chi) &\leq \frac{\sqrt{r}}{2\pi}\max_{x}\Big|\sum_{0<|k|\leq r}\frac{\overline{\chi}(-k)}{k}(e(kx/r)-1)\Big| + O(\log r) \\ &\ll \sqrt{r}\log r. \end{aligned}$$

- We might naively expect the size of $M(\chi)$ to be around $\sqrt{r}$ (roughly "squareroot cancellation").
- **Key point:** If our naive expectation is correct, we only need to think about quite large values of x, because for $x \leq r^{0.55}$ (say) the Burgess bound implies that

$$\Big|\sum_{n\leq x}\chi(n)\Big| \ll (xr^{3/8})^{1/2}\log r \ll r^{0.49}.$$

- So this problem about $M(\chi)$ is a problem about *long* character sums (large x).
- Thus we need to keep the Pólya Fourier expansion (PFE) in mind.
- Thanks to the factor $1/k$ in the PFE, we expect the tails (large k) not to contribute very much. For example, for any *fixed* x (temporarily ignoring the maximum) we have

$$\begin{aligned}
& \frac{1}{r-1}\sum_{\chi \bmod r}\Big|\sum_{r/2<k\leq r}\frac{\overline{\chi}(-k)}{k}(e(kx/r)-1)\Big|^2 \\
= & \frac{1}{r-1}\sum_{\chi \bmod r}\sum_{r/2<k,l\leq r}\frac{\overline{\chi(-k)}}{k}(e(\frac{kx}{r})-1)\frac{\chi(-l)}{l}(e(\frac{-lx}{r})-1) \\
= & \sum_{r/2<k<r}\frac{1}{k^2}|e(\frac{kx}{r})-1|^2 \\
\ll & 1/r.
\end{aligned}$$

Random multiplicative functions

We can use the PFE to (partially) capture the periodic structure of Dirichlet characters. How can we model/investigate the multiplicative structure?

Definition 1

Let $(f(p))_{p \text{ prime}}$ be independent Steinhaus random variables (i.e. distributed uniformly on the unit circle $\{|z| = 1\}$). We define a Steinhaus random multiplicative function *by setting $f(n) := \prod_{p^a || n} f(p)^a$ for all n, where $p^a || n$ means that p^a is the highest power of p that divides n.*

Steinhaus random multiplicative functions are totally multiplicative: $f(mn) = f(m)f(n)$ for all m, n.

The general strategy:

- On n that aren't too large (e.g. $n \leq \sqrt{r}$, away from the point where we see the reflection/Fourier flip property of Dirichlet characters), it seems reasonable to *model* the values $\chi(n)$ (for randomly chosen χ mod r) by a Steinhaus random multiplicative function $f(n)$.
- The values $f(n)$ are **not** all independent, e.g. $f(6) = f(2)f(3)$. So analysing the random multiplicative model can be quite tricky...
- But if we can analyse the random model successfully, we can conjecture things about the character sum problem.
- If the method for analysing the random model is sufficiently robust (or can be made sufficiently robust), we might also be able to prove things in the character sum case.

For example:
If $1 \leq n, m < r$ then

$$\frac{1}{r-1} \sum_{\chi \bmod r} \chi(n)\overline{\chi(m)} = \frac{1}{r-1} \sum_{\chi \bmod r} \chi(n/m) = \mathbf{1}_{n \equiv m \bmod r} = \mathbf{1}_{n=m}.$$

For *any* n, m, we have

$$\mathbb{E} f(n)\overline{f(m)} = \mathbb{E} \prod_{p^a || n} f(p)^a \prod_{p^b || m} f(p)^{-b} = \mathbf{1}_{n=m}.$$

So if $x < r$, and if $(a_n)_{n \leq x}$ are *any* coefficients, then

$$\begin{aligned} \frac{1}{r-1} \sum_{\chi \bmod r} |\sum_{n \leq x} a_n \chi(n)|^2 &= \frac{1}{r-1} \sum_{\chi \bmod r} \sum_{n,m \leq x} a_n \chi(n) \overline{a_m} \overline{\chi(m)} \\ &= \sum_{n \leq x} |a_n|^2 = \mathbb{E} |\sum_{n \leq x} a_n f(n)|^2. \end{aligned}$$

More generally:

For any $k \in \mathbb{N}$ such that $x^k < r$, we have

$$\begin{aligned} \frac{1}{r-1} \sum_{\chi \bmod r} |\sum_{n \leq x} a_n \chi(n)|^{2k} &= \frac{1}{r-1} \sum_{\chi \bmod r} |(\sum_{n \leq x} a_n \chi(n))^k|^2 \\ &= \mathbb{E}|(\sum_{n \leq x} a_n f(n))^k|^2 \\ &= \mathbb{E}|\sum_{n \leq x} a_n f(n)|^{2k}. \end{aligned}$$

We'll come back to this later, and in Lectures 3 and 4.

In Lecture 3, we will also encounter:

Definition 2

Let $(f(p))_{p \text{ prime}}$ *be independent Rademacher random variables (i.e. taking values* ± 1 *with probability* $1/2$ *each). We define a* Rademacher random multiplicative function *by setting* $f(n) := \prod_{p|n} f(p)$ *for all squarefree* n*, and* $f(n) = 0$ *when* n *is not squarefree.*

Rademacher random multiplicative functions are a good model for a randomly chosen Legendre symbol, restricted to squarefree n.

(They were originally introduced to model the Möbius function $\mu(n)$.)

A distributional result

Theorem 1 (Bober, Goldmakher, Granville & Koukoulopoulos, 2018)

Uniformly for all $1 \leq \tau \leq \log\log r - 4$*, we have*

$$e^{-C_1 e^{\tau}/\tau} \leq \frac{1}{r-1}\#\{\chi \ \mathit{mod}\ r : M(\chi) \geq \tau \frac{e^{\gamma}}{\pi}\sqrt{r}\} \leq e^{-C_2 e^{\tau}/\tau},$$

where $C_1, C_2 > 0$ *are absolute constants.*

Theorem 1 improves on various earlier bounds, e.g. Montgomery and Vaughan (1979); Bober and Goldmakher (2013).

We say $M(\chi)$ has "doubly exponential tails".

Key steps in the proof:

- Show that for a sufficiently large proportion (depending on τ) of characters χ mod r, we can restrict the Pólya Fourier expansion (PFE) to numbers k that are $\approx e^{\tau}$-smooth (and of size at most e^{τ^2}, say).
- **Recall:** a number k is y-smooth (or y-friable) if all prime factors of k are $\leq y$.
- Analyse the maximum of the remaining smooth part of the PFE using the random multiplicative model and/or trivial bounds.

First part:
We saw previously that, if we fix some x rather than taking a maximum, then

$$\frac{1}{r-1}\sum_{\chi \bmod r}\Big|\sum_{r/2<k\le r}\frac{\overline{\chi}(-k)}{k}(e(kx/r)-1)\Big|^2 \ll 1/r.$$

To handle smaller k (and deal with the maximum over x), the approach is roughly:

- consider a higher exponent than the square, to boost the saving even for small k;
- small prime factors (repeated with very high multiplicity) cause the bounds to blow up when taking high exponents—this is why we remove the $\approx e^{\tau}$-smooth numbers in advance;
- use the union bound (carefully) to deal with $\max_x$.

Second part:
We now want to analyse the distribution, as χ mod r varies, of

$$\max_x \left|\frac{\tau(\chi)}{2\pi i} \sum_{\substack{0<|k|\leq e^{\tau^2}, \\ k \text{ is } e^{\tau} \text{ smooth}}} \frac{\overline{\chi}(-k)}{k}(e(kx/r)-1)\right|.$$

Almost trivially, this has size

$$\frac{\sqrt{r}}{2\pi} \max_x \left| \sum_{\substack{0<k\leq e^{\tau^2}, \\ k \text{ is } e^{\tau} \text{ smooth}}} \frac{\overline{\chi}(k)}{k}(\overline{\chi}(-1)(e(\frac{kx}{r})-1)-(e(\frac{-kx}{r})-1))\right|$$

$$\lesssim \frac{\sqrt{r}}{2\pi} \sum_{\substack{0<k\leq e^{\tau^2}, \\ k \text{ is } e^{\tau} \text{ smooth}}} \frac{2}{k} \leq \frac{\sqrt{r}}{\pi} \prod_{p\leq e^{\tau}} (1-\frac{1}{p})^{-1}.$$

The Mertens estimate implies that $\prod_{p \le e^\tau}(1-\frac{1}{p})^{-1} \sim e^\gamma \tau$ (as $\tau \to \infty$), so the maximum of the smooth part of the PFE is $\lesssim \frac{\sqrt{r} e^\gamma \tau}{\pi}$ for *all* Dirichlet characters χ mod r. This suffices for the *upper bound* in Theorem 1.

Of course, it isn't just a coincidence that things worked out like this!

The smoothness parameter e^τ was chosen to match the bound we are seeking on $M(\chi)$.

It is reasonable to use trivial bounds for the smooth piece, because the factors $1/k$ in the PFE imply that *the most probable way for* $M(\chi)$ *to be large is a conspiracy of* $\chi(p)$ *for the smallest primes* p.

For the *lower bound* in Theorem 1, we need to prove the existence of sufficiently many χ mod r that make the maximum of the smooth PFE large.

Since we want to produce large values, we can simplify matters by picking a special value of x rather than retaining the maximum. Taking $x = r/2$, and restricting to χ for which $\chi(-1) = -1$, then

$$\begin{aligned}
& \frac{\sqrt{r}}{2\pi}\Big|\sum_{\substack{0<k\leq e^{\tau^2},\\ k \text{ is } e^{\tau} \text{ smooth}}} \frac{\overline{\chi}(k)}{k}\Big(\overline{\chi}(-1)\big(e(\frac{kx}{r})-1\big)-\big(e(\frac{-kx}{r})-1\big)\Big)\Big| \\
= \ & \frac{\sqrt{r}}{2\pi}\Big|\sum_{\substack{0<k\leq e^{\tau^2},\\ k \text{ is } e^{\tau} \text{ smooth}}} \frac{\overline{\chi}(k)}{k}(2-2\cos(\pi k))\Big| \\
= \ & \frac{2\sqrt{r}}{\pi}\Big|\sum_{\substack{0<k\leq e^{\tau^2},\\ k \text{ is } e^{\tau} \text{ smooth}}} \frac{\overline{\chi}(k)}{k}\mathbf{1}_{k \text{ odd}}\Big|.
\end{aligned}$$

Now we *expect* that the proportion of χ for which this is $\geq \tau \frac{e^\gamma}{\pi}\sqrt{r}$ is

$$\begin{aligned}
&\approx \mathbb{P}\left(\frac{2\sqrt{r}}{\pi}\Big|\sum_{\substack{0<k\leq e^{\tau^2},\\ k \text{ is } e^\tau \text{ smooth}}} \frac{f(k)}{k}\mathbf{1}_{k \text{ odd}}\Big| \geq \tau\frac{e^\gamma}{\pi}\sqrt{r}\right) \\
&\approx \mathbb{P}\left(2\Big|\prod_{3\leq p\leq e^\tau}(1-\frac{f(p)}{p})^{-1}\Big| \geq \tau e^\gamma\right) \\
&\approx \mathbb{P}\left(|f(p)-1| \leq \frac{1}{\log\tau}\ \forall\ 3\leq p\leq e^\tau\right) \\
&\approx (\frac{1}{\log\tau})^{e^\tau/\tau},
\end{aligned}$$

as desired (more or less).

Since we are only dealing with values $\chi(p)$ and $f(p)$ for very small p ($\leq e^\tau \leq e^{\log\log r - 4} = \frac{\log r}{e^4}$), it is fairly easy to compare the actual character sums with the random multiplicative model.

For example, the moments

$$\frac{1}{r-1} \sum_{\chi \bmod r} \Big| \sum_{\substack{0<k\leq e^{\tau^2}, \\ k \text{ is } e^\tau \text{ smooth}}} \frac{\overline{\chi}(k)}{k} \mathbf{1}_{k \text{ odd}} \Big|^{2l}, \qquad \mathbb{E}\Big| \sum_{\substack{0<k\leq e^{\tau^2}, \\ k \text{ is } e^\tau \text{ smooth}}} \frac{f(k)}{k} \mathbf{1}_{k \text{ odd}} \Big|^{2l}$$

will exactly agree for all $l \leq (\log r)/\tau^2$.

Using high degree polynomials in $\chi(p)$ or $f(p)$ to approximately detect the conditions $|\chi(p) - 1| \leq \frac{1}{\log\tau}$ and $|f(p) - 1| \leq \frac{1}{\log\tau}$ and taking averages, one can also show the probabilities and proportions roughly agree. (cf. Lecture 4)

Further developments

- Theorem 1 implies that for a very large proportion of χ mod r, we have $M(\chi) \ll \sqrt{r} \log\log r$.
- Assuming the Generalised Riemann Hypothesis is true, Montgomery and Vaughan (1977) proved that $M(\chi) \ll \sqrt{r} \log\log r$ for *all* $\chi \neq \chi_0$ mod r. This is best possible, in view of the lower bound in Theorem 1 (or earlier results).
- But this bound can be improved if one assumes more (about the order and/or "pretentiousness") of the character χ.

- There are several other interesting problems about long character sums $\sum_{n \leq x} \chi(n)$ (i.e. with large x), which again reduce to analysing only the first terms in the PFE.
- For example, Hussain (2020) investigates the path distribution of $t \mapsto \frac{1}{\sqrt{r}} \sum_{n \leq rt} \chi(n)$.
- Thanks to the factor $1/k$ in the PFE, one can solve these problems without very precise information about random multiplicative functions.