Notes

domenica 27 giugno 2021 16:17

A weighted one-level density of the non-trivial zeros of L-functions

The density conjecture

- $\mathcal{F}$ a "natural" family of L-functions, ordered by log-conductor $c(L)$ with symmetry type (in the sense of Katz-Sarnak) G
- Assume RH: $\rho_L = \frac{1}{2} + i\gamma_L$ $(\gamma \in \mathbb{R})$ the non-trivial zeros of $L \in \mathcal{F}$
- f a test function ($f \approx \mathbb{1}_{[0,1]}$, smoothing)

Conj (K-S)

$$\frac{1}{\#\mathcal{F}} \sum_{L \in \mathcal{F}} \sum_{\gamma_L} f(c(L)\gamma_L) \xrightarrow{\#\mathcal{F} \to \infty} \int_{-\infty}^{+\infty} f(x) W_G(x)\, dx$$

ex $\{\zeta(s+it) : t \in \mathbb{R}\}$ unitary U

$\{L(\frac{1}{2}, \chi_d) : d>0, f.d.\}$ symplectic USp

$\{L_\Delta(\frac{1}{2}, \chi_d) : d>0, f.d.\}$ even orthogonal SO^+

A weighted 1-LD

We tilt the average over $\mathcal{F}$, multiplying by $L(\frac{1}{2})^k$

$k \in \mathbb{N}$

$$D_k^{\mathcal{F}}(f) := \frac{1}{\sum_{L \in \mathcal{F}} V(L(\frac{1}{2}))^k} \sum_{L \in \mathcal{F}} \sum_{\gamma_L} f(c(L)\gamma_L)\, V(L(\tfrac{1}{2}))^k$$

$$\left[\; V(z) = \begin{cases} |z|^2 & U \\ z & USp/SO^+ \end{cases} \;\right]$$

This quantity is a special case of

$$\sum_{L \in \mathcal{F}} g(L)\, V(L(\tfrac{1}{2}))^k$$

ex Unitary case

2k-th moment comes from those L-fs such that $|L(\frac{1}{2})| \approx (\log X)^{k+o(1)}$ which form a thin subset of size $\#\mathcal{F} / (\log X)^{k^2+o(1)}$

Then if g is bounded, we are focussing on the L-fs such that

$$(\log X)^{k-\varepsilon} \ll |L(\tfrac{1}{2})| \ll (\log X)^{k+\varepsilon}$$

i.e. the ones which are responsible for the 2k-th moment

IDEA The structure is the same as suggested by the density conjecture:

$$D_k^{\mathcal{F}}(f) = \int_{-\infty}^{+\infty} f(x)\, W_G^k(x)\, dx + \dots$$

THM Assume RH & RC. For three specific families (with different symmetry types), $k \le 4$, then $(*)$ is a thm.

[C-S]

THM Unconditionally, with $k=1$, for the continuous family $\{\zeta(s+it) : t \in \mathbb{R}\}$ then $(*)$ is a thm.

[H-R / BCHB]

Conjectures / Speculations

① There are relations between the $U/USp/SO^+$ weighted kernels

$$W_{SO^+}^k(x) = W_{USp}^{k-1}(x) \qquad\qquad W_U^k(x) = \frac{W_{USp}^k(x) + W_{SO^+}^k(x)}{2}$$

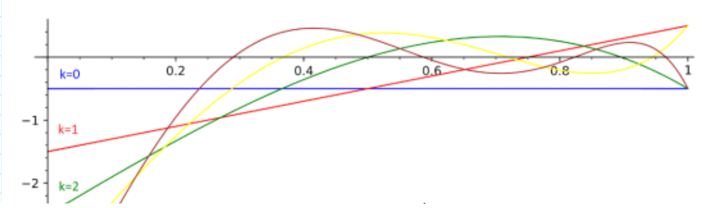
(DELAY) (AVERAGE)

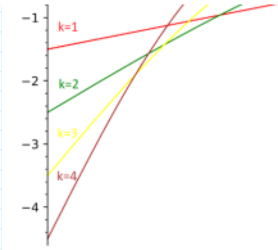
② • $\widehat{W}_{USp}^k(y) = \delta_0(y) + \chi_{[-1,1]}(y)\, P_{USp}^k(|y|)$ → poly

degree $2k-1$

$P_{USp}^k(0) = -\dfrac{2k+1}{2}$

$P_{USp}^k(1) = \dfrac{(-1)^{k+1}}{2}$





$P^k_{USp}(y)$ $\qquad$ $P^k_{USp}(1) = \dfrac{(-1)^{k+1}}{2}$

$$\bullet\ W^k_{USp}(x) = 1-(2k+1)\frac{\sin(2\pi x)}{2\pi x} + \sum_{J=1}^{k} \frac{k(k+1)}{2^{2J-2}\pi^{2J-1}} \frac{c_{J,k}}{2J-1} \frac{d^{2J-1}}{dx^{2J-1}}\left[\frac{1-\cos(2\pi x)}{2\pi x}\right]$$

with $c_{J,k} = \frac{1}{J}\binom{k-1}{J-1}\binom{k+J}{J-1}$

$$W^0_{USp}(x) := W_{USp}(x) = 1-\frac{\sin(2\pi x)}{2\pi x},$$
$$W^1_{USp}(x) := 1+\frac{\sin(2\pi x)}{2\pi x}-\frac{2\sin^2(\pi x)}{(\pi x)^2},$$
$$W^2_{USp}(x) := 1-\frac{\sin(2\pi x)}{2\pi x}-\frac{24(1-\sin^2(\pi x))}{(2\pi x)^2}+\frac{48\sin(2\pi x)}{(2\pi x)^3}-\frac{96\sin^2(\pi x)}{(2\pi x)^4},$$
$$W^3_{USp}(x) := 1+\frac{\sin(2\pi x)}{2\pi x}-\frac{12\sin^2(\pi x)}{(\pi x)^2}-\frac{240\sin(2\pi x)}{(2\pi x)^3} - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4}+\frac{2880\sin(2\pi x)}{(2\pi x)^5}-\frac{90\sin^2(\pi x)}{(\pi x)^6},$$
$$W^4_{USp}(x) := 1-\frac{\sin(2\pi x)}{2\pi x}-\frac{10(1+\cos(2\pi x))}{(\pi x)^2}+\frac{90\sin(2\pi x)}{(\pi x)^3} - \frac{15(3-31\cos(2\pi x))}{(\pi x)^4}-\frac{1470\sin(2\pi x)}{(\pi x)^5} - \frac{315(1+9\cos(2\pi x))}{(\pi x)^6}+\frac{3150\sin(2\pi x)}{(\pi x)^7}-\frac{1575(1-\cos(2\pi x))}{(\pi x)^8}$$

③ At $x=0$ $\qquad W^k_{USp}(x) \xrightarrow{x\to 0} 0$ of order increasing with k

Indeed $\quad W^k_{USp}(x) = \sum_{m=1}^{\infty} \beta_{m,k}\, x^{2m}$ $\quad$ with $\beta_{m,k}$ explicit
(in terms of hypergeometric functions)

In particular

$$W^k_{USp}(x) \sim \frac{2\pi^{2k+2}}{(2k+1)!!\,(2k+3)!!}\, x^{2k+2}$$

REM $\quad$ This kind of weight appears naturally in a work by Kowalski-Saha-Tsimerman

Given a Siegel modular form F of genus 2, they compute the one level density of the spinor L-f of F with weight

$$\omega^F \approx \left|\begin{matrix}\text{first F. coeff of}\\ \text{the modular form}\end{matrix}\right|^2$$

They expected an orthogonal symmetry type but got symplectic!

Why ?

Böcherer conj : $\omega^F \simeq L(\frac{1}{2}, F)$

proportional

$$\left\{ \begin{array}{l} \text{From above :} \\ W^1_{SO^+}(x) = W_{USp}(x) = 1 - \dfrac{\sin(2\pi x)}{2\pi x} \end{array} \right\}$$